# Practical motivation for dependent extremes

Often, several variables are stochastically dependent,
for example in environmental and climatic data.

**Examples:**

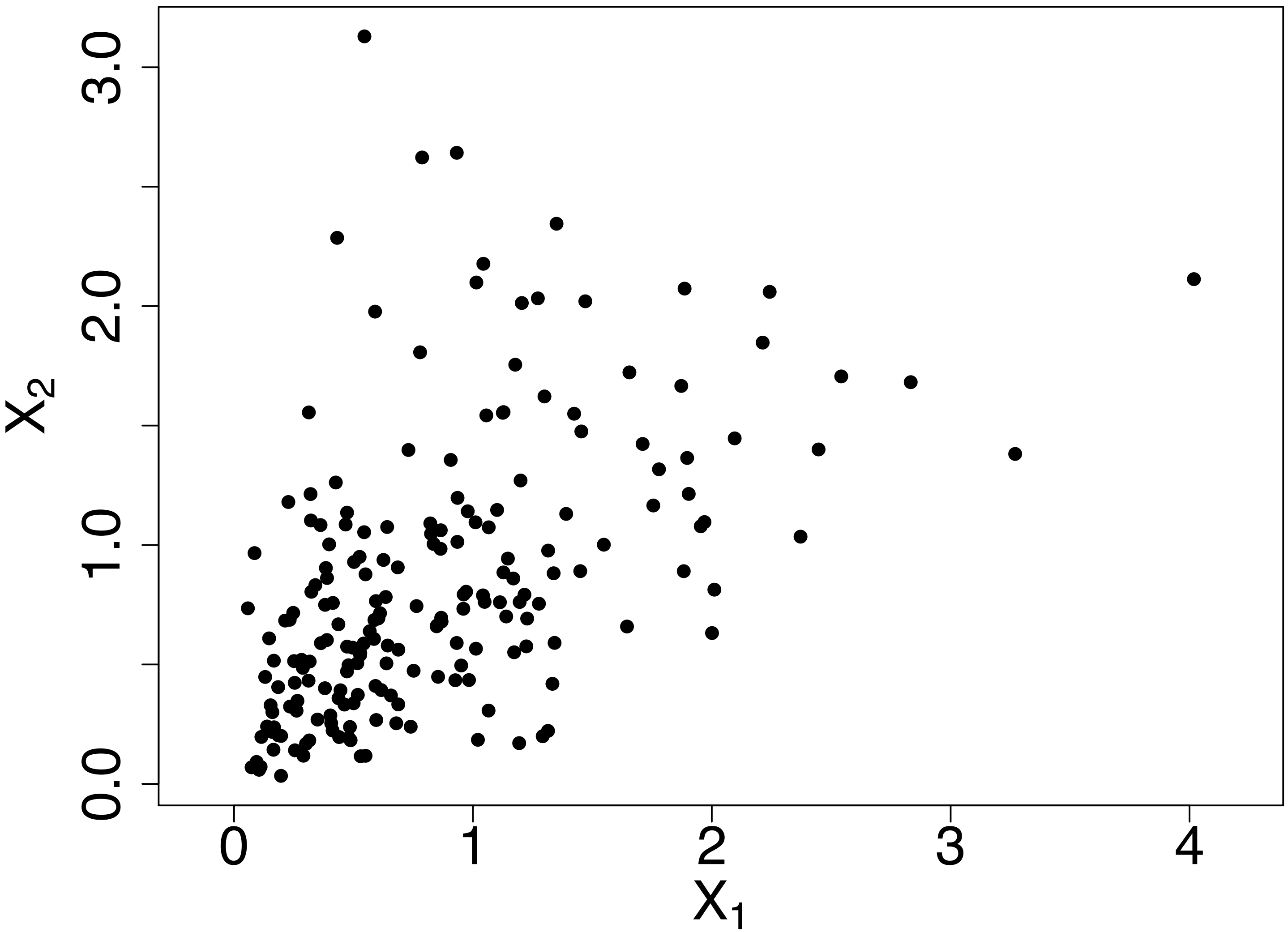
- Different physical variables observed at the same location, such as minimum temperature, maximum temperature, precipitation, wind speed.
- The same physical variable observed at different locations, such as precipitation at different locations of a river catchment.

**Many interesting aspects of dependent extremes:**

- **Aggregation** of extreme observations in several components (example: cumulated precipitation $\Rightarrow$ flood risk)
- **Spatial extent** and **temporal duration** of environmental extreme events
- **Reliability:** simultaneous failure of several critical components

# Illustration: a bivariate sample with dependence

Scatterplot of an iid bivariate sample $\boldsymbol{X}_i = (X_{i,1}, X_{i,2})$, $i = 1, 2, \ldots, n$.



## A note on notations (multivariate / process)

Representations for extremes of random vectors and stochastic processes are structurally quite similar.

For indexing the variables of interest,

- we can either put focus on the multivariate aspect and use indices $1, \dots, d$ for the $d$ components of a random vector

  $$(X_1, \dots, X_d)$$

  (and we can write $D = \{1, \dots, d\}$ for the domain),
- or we put focus on the process aspect (for example, when working with a random field on a nonempty domain $D \subset \mathbb{R}^k$) and use notation such as

  $$\{X(s),\ s \in D\}$$

  for the whole process, or

  $$(X(s_1), \dots, X(s_d))$$

  for the multivariate vector of variables observed at $d$ locations $s_1, \dots, s_d \subset \mathbb{R}^k$.

When the distinction is important, we point it out explicitly (for example, for "functional convergence" in a space of functions with continuous sample paths defined over a compact domain $D$).

## Componentwise maxima of random vectors

Consider a sequence of iid random vectors

$$\boldsymbol{X}_i = (X_{i,1}, \ldots, X_{i,d}) \stackrel{d}{=} \boldsymbol{X} \sim F_{\boldsymbol{X}},$$

where $F_{\boldsymbol{X}}$ is the joint distribution of the components of $\boldsymbol{X}$:

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = F_{\boldsymbol{X}}(x_1, \ldots, x_d) = \Pr(X_1 \leq x_1, \ldots, X_d \leq x_d)$$

The **componentwise maximum**

$$\boldsymbol{M}_n = (M_{n,1}, \ldots, M_{n,d}) = \left( \max_{i=1}^{n} X_{i,1}, \ldots, \max_{i=1}^{n} X_{i,d} \right)$$

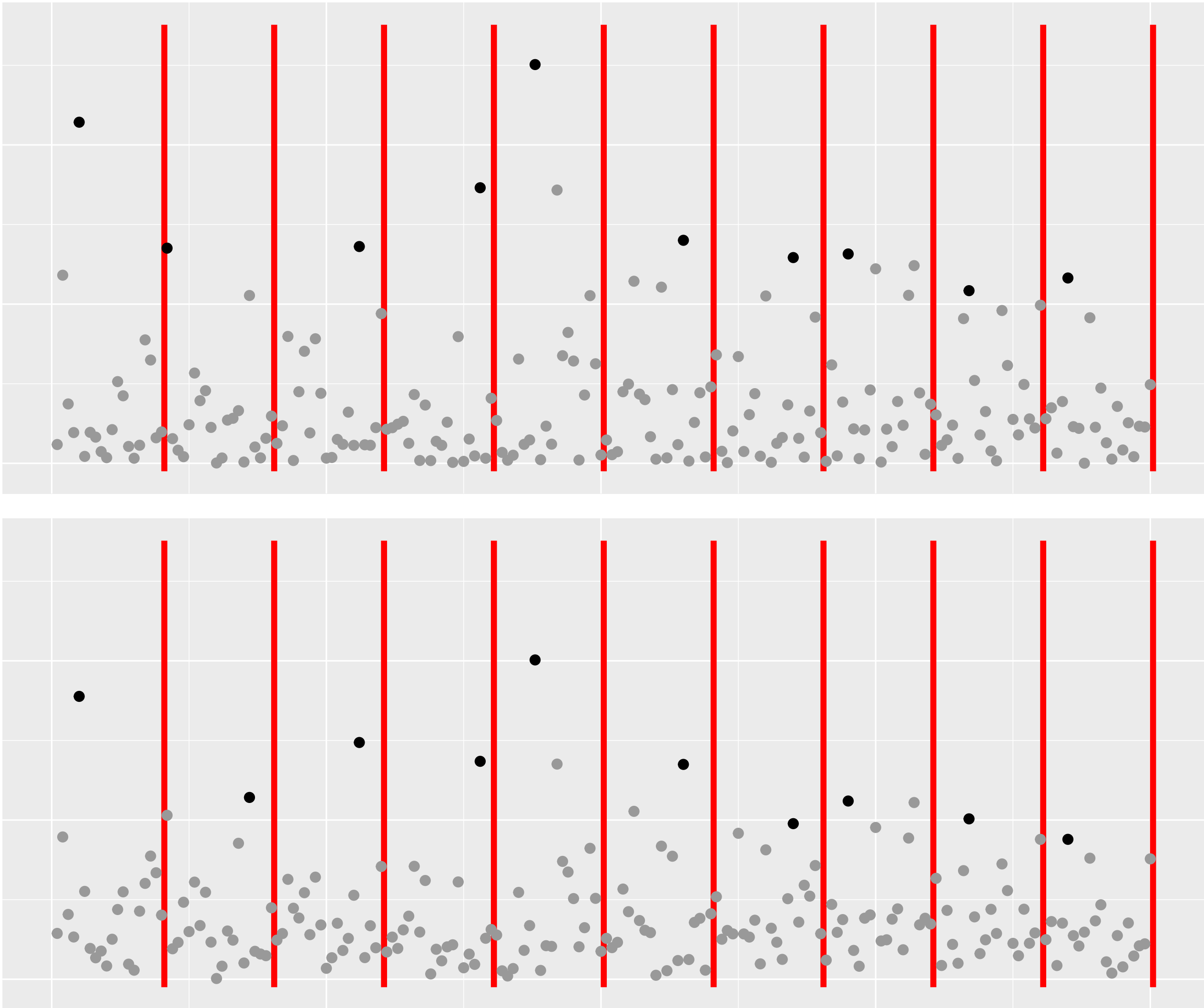
has distribution $F_{\boldsymbol{X}}^n$, that is, for $\boldsymbol{x} = (x_1, \ldots, x_d)$,

$$F_{\boldsymbol{X}}^n(\boldsymbol{x}) = (F_{\boldsymbol{X}}(\boldsymbol{x}))^n = \Pr(X_{i,1} \leq x_1, \ldots, X_{i,d} \leq x_d,\ i = 1, \ldots, n)$$

⚠ The componentwise maximum $\boldsymbol{M}_n$ can be composed of values $X_{i,j}$ with different indices $i$.

# Illustration: bivariate componentwise block maxima

A bivariate series $\boldsymbol{X}_i = (X_{i,1}, X_{i,2})$ (with strong cross-correlation) and its componentwise maxima within the blocks separated by red lines. Most but not all of the maxima occur at the same time in the two series.



## Definition: max-stable distribution; max-stable process

A **multivariate ($d$-dimensional) distribution** $G$ is called **max-stable** if there exist deterministic vector sequences $\boldsymbol{\alpha}_n = (\alpha_{n,1}, \ldots, \alpha_{n,d})$ and $\boldsymbol{\beta}_n = (\beta_{n,1}, \ldots, \beta_{n,d}) > \mathbf{0}$, $n \in \mathbb{N}$, such that

$$G^n(\boldsymbol{\alpha}_n + \boldsymbol{\beta}_n \boldsymbol{z}) = G(\boldsymbol{z}), \quad \boldsymbol{z} \in \mathbb{R}^d.$$

If all finite-dimensional distributions of a stochastic process $\boldsymbol{Z} = \{Z(s), s \in D \subset \mathbb{R}^k\}$ are max-stable, we call $\boldsymbol{Z}$ a **max-stable process**.

Equivalently, if $\boldsymbol{X}_1 \sim G$, then the componentwise maximum over $n$ iid copies of $\boldsymbol{X}_1$ satisfies

$$\frac{\boldsymbol{M}_n - \boldsymbol{\alpha}_n}{\boldsymbol{\beta}_n} \stackrel{d}{=} \boldsymbol{X}_1, \quad n \in \mathbb{N}.$$

⚠ Multivariate max-stability is stronger than max-stability of the univariate marginal distributions.

- If $\boldsymbol{Z} = (Z_1, \ldots, Z_d) \sim G$ with $Z_j \sim G_j$, then the univariate marginal distributions $G_j$ are max-stable:

$$Gj(z_j) = \mathrm{GEV}(z_j; \xi_j, \mu_j, \sigma_j) = \Pr(Z_j \leq z_j) = G(\infty, \ldots, \infty, z_j, \infty, \ldots, \infty).$$

- Additionally, max-stability of $G$ implies a stability property for the dependence structure.

# Multivariate Maximum-Domain-of-Attraction theorem

**Theorem: Multivariate Maximum Domain of Attraction**

If there exist deterministic normalizing vector sequences $\boldsymbol{a}_n = (a_{n,1}, \ldots, a_{n,d})$ and $\boldsymbol{b}_n = (b_{n,1}, \ldots, b_{n,d}) > 0$, $n \in \mathbb{N}$, such that the following convergence holds,

$$\frac{\boldsymbol{M}_n - \boldsymbol{a}_n}{\boldsymbol{b}_n} \to \boldsymbol{Z} = (Z_1, \ldots, Z_d) \sim G, \quad n \to \infty,$$

where $\boldsymbol{Z}$ has non-degenerate marginal distributions, then $G$ is a **multivariate extreme-value distribution**, that is, a **multivariate max-stable distribution**.

If all finite-dimensional distributions of a process $\boldsymbol{X} = \{X(s),\ s \in D \subset \mathbb{R}^k\}$ satisfy the above convergence, then $\boldsymbol{Z} = \{Z(s), s \in D \subset \mathbb{R}^k\}$ is a **max-stable process**.

(see, for instance, Resnick (1987) for the proof)

**Remark:** For stochastic processes, we here define convergence in terms of finite-dimensional distributions. There also exist results for convergence in spaces of continuous functions over a compact domain $D$.

## Formulation using standardized marginal distributions

To focus on the extremal dependence structure, it is useful to **standardize the marginal distributions** $F_j$ of $X_j$ and $G_j$ of $Z_j$.

- Often, the **unit Fréchet marginal distribution** is used:

$$G_j^\star(z) = \mathrm{GEV}(z; \xi = 1, \mu = 1, \sigma = 1) = \exp\left(-\frac{1}{z}\right), \quad z > 0.$$

- We can transform any continuous random variable $X \sim F$ towards a variable with unit Fréchet distribution as follows: $X^\star = -\frac{1}{\log F(X)} \sim G^\star$.

- If $X_j \sim \mathrm{GEV}(\xi, \mu, \sigma)$, then $X_j^\star = \left(1 + \xi \frac{X-\mu}{\sigma}\right)^{1/\xi} \sim G_j^\star$.

- If $G$ is a multivariate max-stable distribution, we write $G^\star$ for the corresponding max-stable distribution with unit Fréchet margins. We call $G^\star$ a **simple max-stable distribution**.

**We call representations simple if they are based on the marginal $\star$-scale.**

# Simple Maximum Domain of Attraction

We use the following notation: $T_{\xi,\mu,\sigma}(z) = \left(1 + \xi \frac{z-\mu}{\sigma}\right)^{1/\xi}$.

## Maximum Domain of Attraction using standardized marginal distributions

Consider a random vector $\boldsymbol{X} \sim F_{\boldsymbol{X}}$. The following two statements are equivalent:

1. The distribution $F_{\boldsymbol{X}}$ is in the MDA of a multivariate max-stable distribution $G$.
2. The following two properties hold jointly:
   1. **Marginal convergence:** Each component $X_j$ is in the univariate MDA of a $\mathrm{GEV}(\xi_j, \mu_j, \sigma_j)$ distribution.
   2. **Convergence on the standardized scale:** The distribution of the marginally standardized random vector

$$\boldsymbol{X}^\star = (X_1^\star, \ldots, X_d^\star) \sim F_{\boldsymbol{X}^\star}$$

satisfies

$$F^n_{\boldsymbol{X}^\star}(n\,\boldsymbol{z}) \to G^\star(\boldsymbol{z}), \quad n \to \infty,$$

i.e., $F_{\boldsymbol{X}^\star}$ is in the MDA of $G^\star$, where

$$G(z_1, \ldots, z_d) = G^\star(T_{\xi_1,\mu_1,\sigma_1}(z_1), \ldots, T_{\xi_d,\mu_d,\sigma_d}(z_d)).$$

With standardized marginal distributions, we can choose normalizing vector sequences $\boldsymbol{a}_n^\star = (0, \ldots, 0)$ and $\boldsymbol{b}_n^\star = (n, \ldots, n)$.

# Some remarks about max-stable dependence

- There is no exhaustive representation of all simple max-stable distributions $G^\star$ using a finite number of parameters.

- We can write $G^\star$ using the **exponent function** $V^\star$,

$$G^\star(\boldsymbol{z}) = \exp(-V^\star(\boldsymbol{z})), \quad \boldsymbol{z} > \boldsymbol{0},$$

  where $t \times V^\star(t\boldsymbol{z}) = V^\star(\boldsymbol{z})$ (**$(-1)$-homogeneity**).

- We say that two variables $X_1$ and $X_2$ are **asymptotically independent** if

$$G(z_1, z_2) = G_1(z_1) \times G_2(z_2),$$

  and in this case

$$G^\star(z_1, z_2) = \exp(-(1/z_1 + 1/z_2)) = \exp(-1/z_1) \times \exp(-1/z_2), \quad z_1, z_2 > 0.$$

# Example: multivariate logistic distribution

A large variety of **parametric multivariate max-stable distribution** has been proposed.

The **multivariate logistic model** was introduced by Emil J. Gumbel in 1960 and can be defined through its exponent function

$$V^\star(\boldsymbol{z}) = \left(z_1^{-1/\alpha} + \ldots + z_d^{-1/\alpha}\right)^\alpha, \quad \boldsymbol{z} > \boldsymbol{0},$$

such that

$$G^\star(z_1, \ldots, z_d) = \exp\left(-\left(z_1^{-1/\alpha} + \ldots + z_d^{-1/\alpha}\right)^\alpha\right), \quad \boldsymbol{z} > \boldsymbol{0}$$

with parameter $0 < \alpha \leq 1$ and

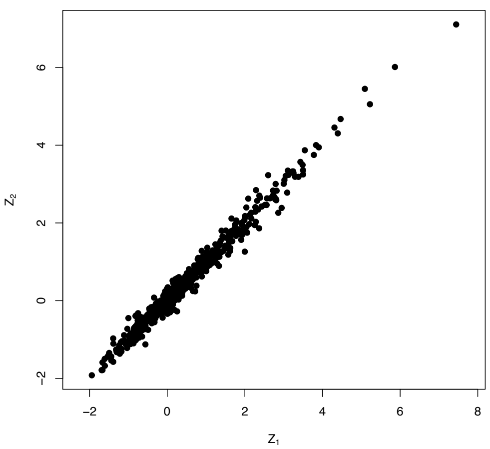
- perfect dependence for $\alpha \to 0$;
- independence for $\alpha = 1$.

# Example: Simulations of bivariate logistic distribution

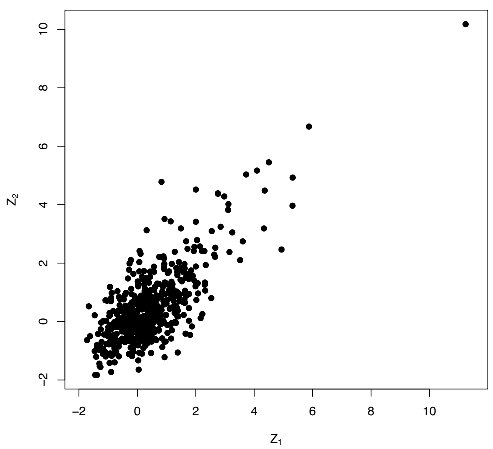
Sample size $n = 500$

Bivariate scatterplots show $\log \boldsymbol{Z}^\star$ (standard Gumbel margins) with $\boldsymbol{Z}^\star \sim G^\star$

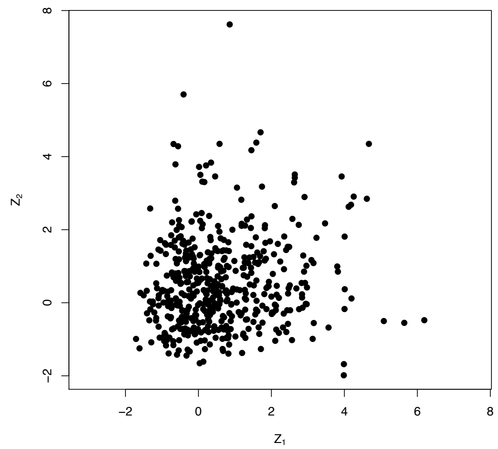
$\alpha = 0.1$




$\alpha = 0.5$




$\alpha = 0.9$



# Example: Huesler–Reiss distribution

**Huesler–Reiss distributions** are related to multivariate Gaussian distributions.
Consider a multivariate Gaussian vector $\tilde{\boldsymbol{Y}}$.

**Bivariate case:** the simple max-stable distribution has parameter $\gamma_{12} = \mathrm{Var}(\tilde{Y}_2 - \tilde{Y}_1) > 0$ and for $z_1, z_2 > 0$,

$$G^\star(z_1, z_2) = \exp\left(-\frac{1}{z_1}\Phi\left(\frac{\sqrt{\gamma_{12}}}{2} + \frac{1}{\sqrt{\gamma_{12}}}\log\frac{z_2}{z_1}\right) - \frac{1}{z_2}\Phi\left(\frac{\sqrt{\gamma_{12}}}{2} + \frac{1}{\sqrt{\gamma_{12}}}\log\frac{z_1}{z_2}\right)\right)$$

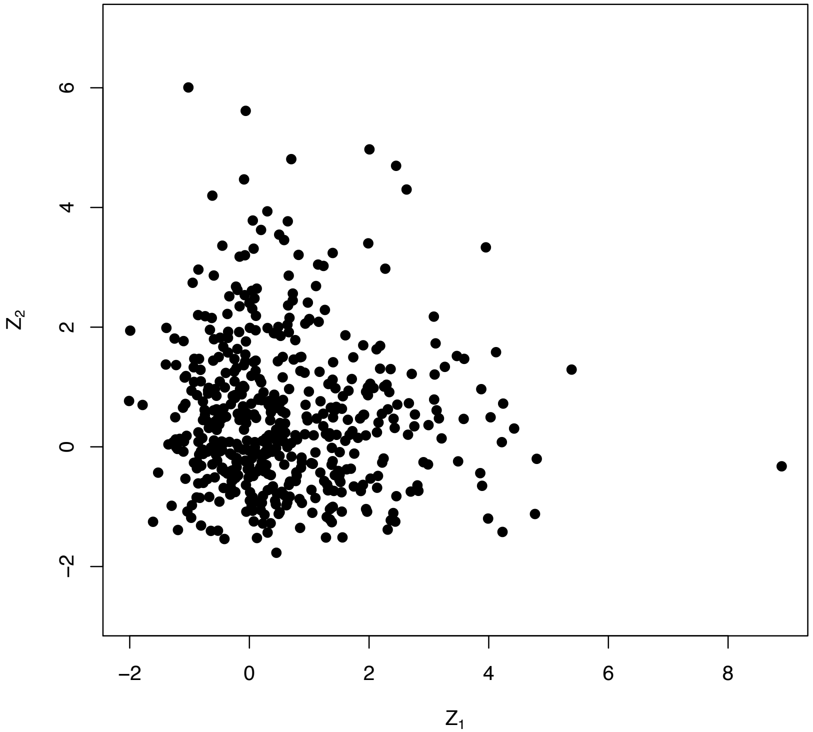
(with standard Gaussian cdf $\Phi$)
$\Rightarrow$ independence for $\gamma_{12} \to \infty$, perfect dependence for $\gamma_{12} \to 0$

The general multivariate distribution $G^\star$ is parametrized by $d(d-1)/2$ variogram values $\gamma_{j_1,j_2} = \mathrm{Var}(\tilde{Y}_{j_2} - \tilde{Y}_{j_1})$ for $1 \le j_1 < j_2 \le d$.

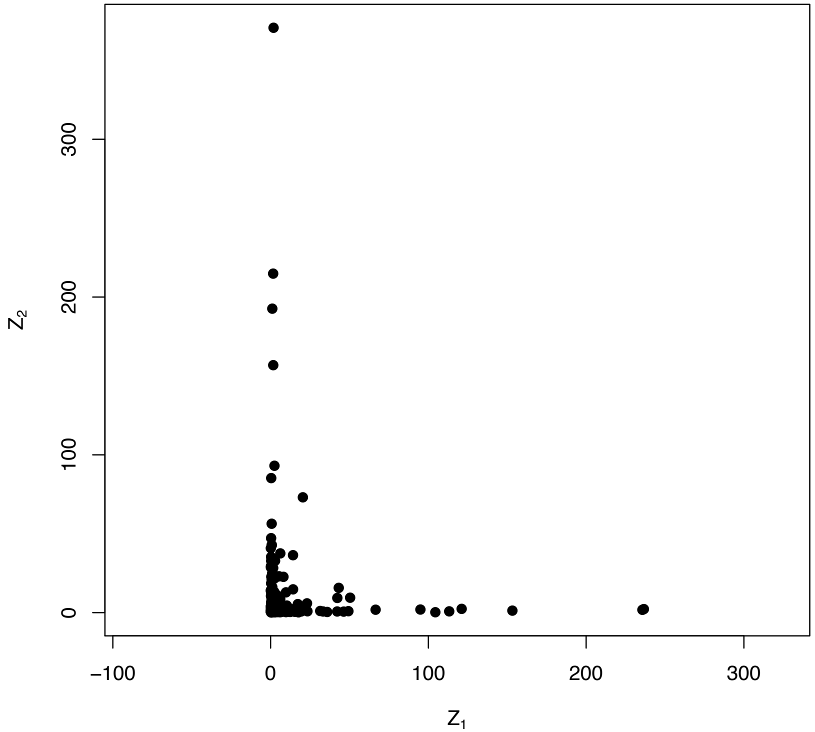
# Example: Simulations of the Huesler–Reiss distribution

Sample size $n = 500$
Relatively weak dependence

log $\boldsymbol{Z}^\star$ (Gumbel margins)



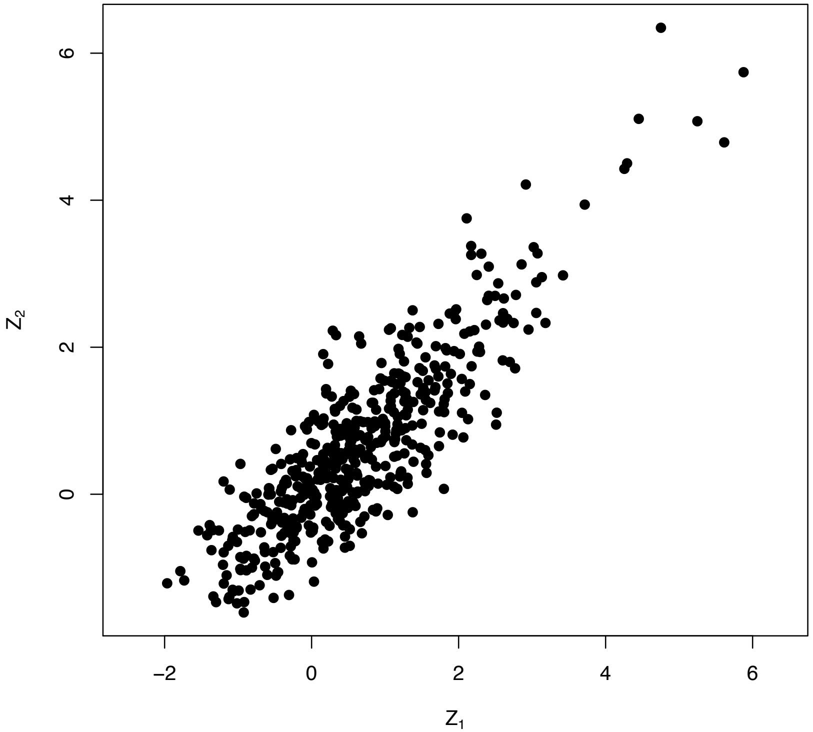

$\boldsymbol{Z}^\star$ (Fréchet margins)



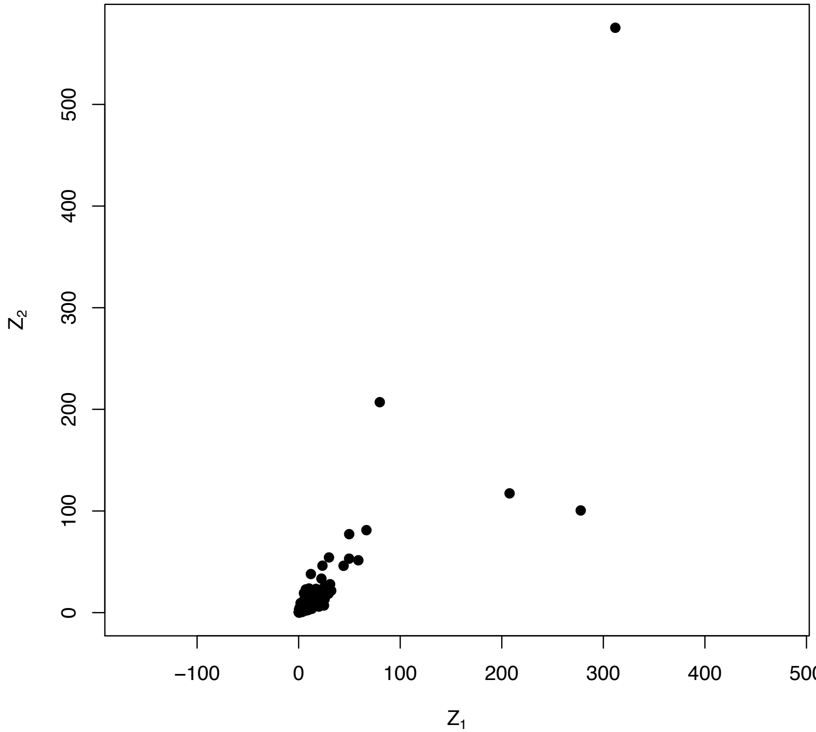
# Example, cont'd

Sample size $n = 500$
Relatively strong dependence

$\log \boldsymbol{Z}^\star$ (Gumbel margins)




$\boldsymbol{Z}^\star$ (Fréchet margins)



## Theorem (Point-process convergence)

For i.i.d. copies $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ of a random vector $\boldsymbol{X} = (X_1, \ldots, X_d) \sim F$, the following two statements are equivalent:

1. The distribution $F$ is in the multivariate MDA of the max-stable distribution $G$ for the normalizing sequences $\boldsymbol{a}_n \in \mathbb{R}^d$ and $\boldsymbol{b}_n > \boldsymbol{0}$.
2. For the normalizing sequences $\boldsymbol{a}_n \in \mathbb{R}^d$ and $\boldsymbol{b}_n > \boldsymbol{0}$, we have the following point-process convergence with a locally finite Poisson point process limit:

$$\left\{\frac{\boldsymbol{X}_i - \boldsymbol{a}_n}{\boldsymbol{b}_n},\ i = 1, \ldots, n\right\} \to \{\boldsymbol{P}_i,\ i \in \mathbb{N}\} \sim \mathrm{PPP}(\Lambda), \quad n \to \infty,$$

   with intensity measure $\Lambda$.

If 1) and 2) hold, then $G(\boldsymbol{z}) = \exp(-V(\boldsymbol{z}))$ with

$$V(\boldsymbol{z}) = \Lambda\left((-\boldsymbol{\infty}, \boldsymbol{z}]^C\right),$$

where the **exponent measure** $\Lambda$ is defined on $A_\Lambda = \left(\overline{A}_{\xi_1,\mu_1,\sigma_1} \times \ldots \times \overline{A}_{\xi_d,\mu_d,\sigma_d}\right) \setminus \boldsymbol{u}_\star$, with the marginal GEV parameters $\xi_j, \mu_j, \sigma_j$, $j = 1, \ldots, d$, where the lower endpoint

$$\boldsymbol{u}_\star = \left(\inf A_{\xi_1,\mu_1,\sigma_1}, \ldots, \inf A_{\xi_d,\mu_d,\sigma_d}\right)$$

is excluded.

# Simple representation with standardized margins

Specifically, the convergence of componentwise maxima and of point patterns is equivalent on the simple scale using standardized marginal distributions in $\boldsymbol{X}^\star$.

## Recall: Standardized marginal scale

- $X_j^\star = -1/\log F_j(X_j)$ (or any other probability integral transform ensuring $X_j^\star \geq 0$ and $x \times \Pr(X_j^\star > x) \to 1$ as $x \to \infty$)
- Normalizing sequences on standardized scale are $\boldsymbol{a}_n = \boldsymbol{0}$ and $\boldsymbol{b}_n = (n, \ldots, n)$
- GEV margins of $G^\star$ are unit Fréchet $G_j^\star(z_j) = \exp(-1/z_j)$, $z_j > 0$ ($\xi_j = 1$, $\mu_j = 1$, $\sigma_j = 1$).

## Simple exponent measure and homogeneity (asymptotic stability)

For any Borel set $B \subset A_\Lambda$, the **simple exponent measure $\Lambda^\star$** satisfies

$$\Lambda(B) = \Lambda^\star(B_{\boldsymbol{\xi},\boldsymbol{\mu},\boldsymbol{\sigma}})$$

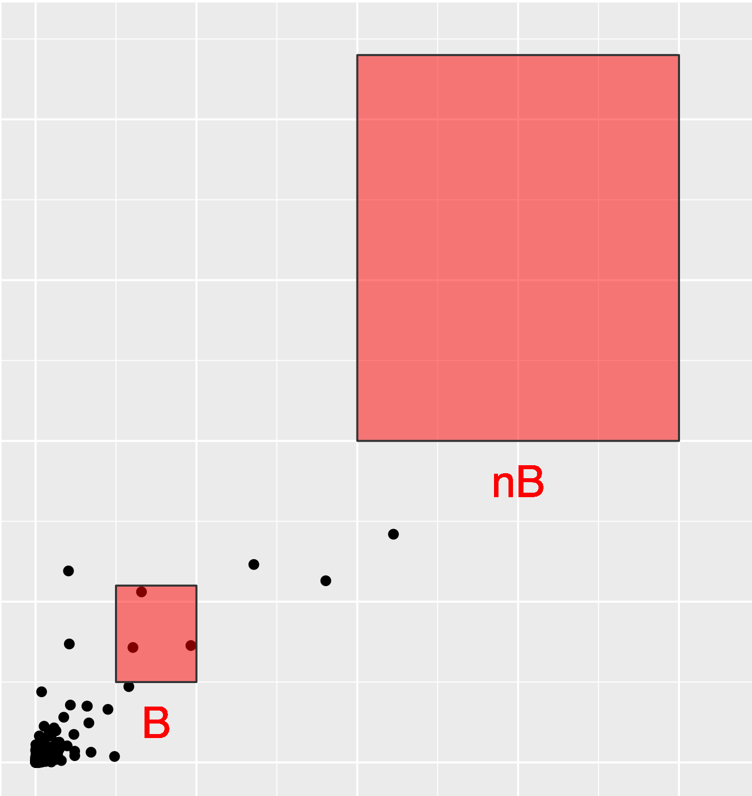
where $B_{\boldsymbol{\xi},\boldsymbol{\mu},\boldsymbol{\sigma}} = \left\{\left(T_{\xi_1,\mu_1,\sigma_1}(x_1), \ldots, T_{\xi_d,\mu_d,\sigma_d}(x_d)\right) \mid (x_1, \ldots, x_d) \in B\right\}$. The simple measure $\Lambda^\star$ is defined on $A_{\Lambda^\star} = [0, \infty)^d \setminus \boldsymbol{0}$ and is **$(-1)$-homogeneous**, that is, for any Borel set $B \subset A_{\Lambda^\star}$, we have

$$t \times \Lambda^\star(tB) = \Lambda^\star(B), \quad t > 0.$$

# Bivariate illustration of asymptotic stability $(D = \{1, 2\})$

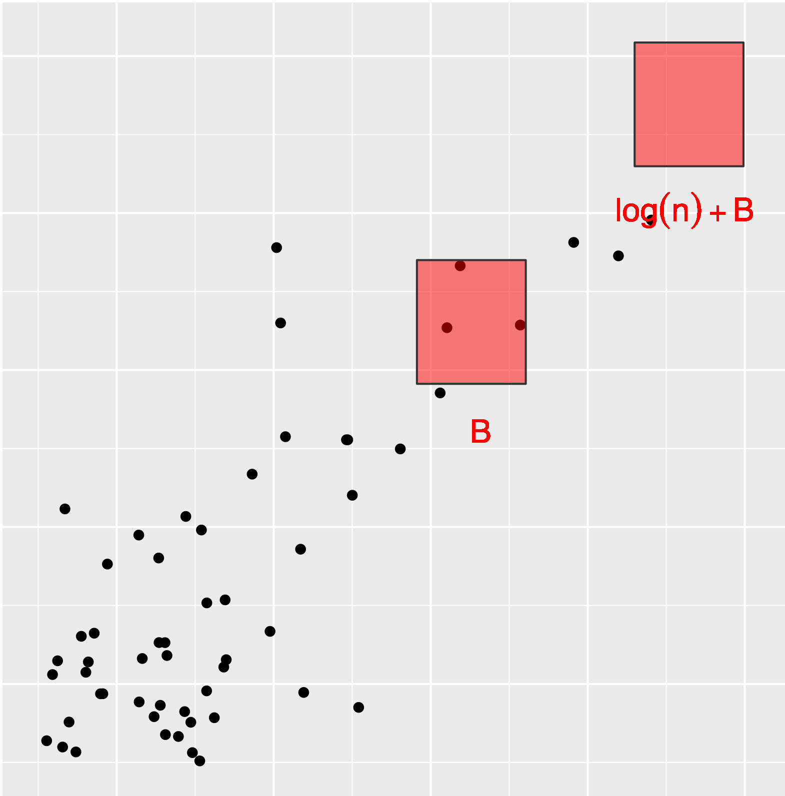
**Simple scale**
$(\boldsymbol{\xi} = (1, 1), \boldsymbol{\mu} = (1, 1), \boldsymbol{\sigma} = (1, 1))$

$\boldsymbol{\alpha}_n = (n, n)$, $\boldsymbol{\beta}_n = (0, 0)$
$n \times \Lambda^\star(nB) = \Lambda^\star(B)$




**Standard exponential scale**
$(\boldsymbol{\xi} = (0, 0), \boldsymbol{\mu} = (0, 0), \boldsymbol{\sigma} = (1, 1))$

$\boldsymbol{\alpha}_n = (1, 1)$, $\boldsymbol{\beta}_n = (\log n, \log n)$
$n \times \Lambda(\log(n) + B) = \Lambda(B)$



# The trinity: three classical multivariate formulations

- The trinity of the three classical limit also holds in the multivariate setting.
- For **threshold exceedances**, a standard approach is to condition on an exceedance in at least one of the $d$ components.
- To avoid technical notation, we focus on the **simple setting**.

## Theorem

**The following three convergences are equivalent:**

- Point-process convergence:

$$\left\{ \frac{\boldsymbol{X}_i^\star}{n}, \ i = 1, \ldots, n \right\} \to \{\boldsymbol{P}_i^\star, \ i \in \mathbb{N}\} \sim \mathrm{PPP}(\Lambda^\star), \quad n \to \infty.$$

- Convergence of componentwise maxima:

$$\frac{\boldsymbol{M}_n^\star}{n} \to \boldsymbol{Z}^\star \sim G^\star, \quad n \to \infty,$$

with $G^\star(\boldsymbol{z}) = \exp(-V^\star(\boldsymbol{z}))$ where $V^\star(\boldsymbol{z}) = \Lambda^\star \left([\boldsymbol{0}, \boldsymbol{z}]^C\right)$.

- Peaks-Over-Threshold convergence:

$$\frac{\boldsymbol{X}^\star}{u} \mid \left( \max_{j=1}^{d} X_j^\star > u \right) \to \boldsymbol{Y}^\star \sim \frac{\Lambda^\star( \cdot \cap [\boldsymbol{0}, \boldsymbol{1}]^C)}{\Lambda^\star \left([\boldsymbol{0}, \boldsymbol{1}]^C\right)}, \quad u \to \infty.$$

# Remarks about the functional setting

- The trinity of limits also holds in the functional setting (e.g., Dombry & Ribatet, 2016).
- Usually one considers $\boldsymbol{X} \in \mathcal{C}(D)$ with compact domain $D$.
- One has to appropriately define weak convergence in a Banach function space.

# The spectral construction of simple processes

## Spectral representation of simple point processes

Any Poisson point process $\{\boldsymbol{P}_i^\star,\ i \in \mathbb{N}\}$ with simple ((−1)-homogeneous) intensity measure $\boldsymbol{\Lambda}^\star$ can be constructed as follows:

$$\{P_i^\star(s),\ i \in \mathbb{N}\} = \{R_i W_i(s),\ i \in \mathbb{N}\}$$

where $R_i = 1/U_i$ and

- $0 < U_1 < U_2 < ....$ are the points of a unit-rate Poisson process on $[0, \infty)$, and
- $\boldsymbol{W}_i = \{W_i(s)\}$ are iid nonnegative random functions, independent of $\{U_i\}$, with $\mathbb{E}W_i(s) = 1$ and $\mathbb{E}W_i(s)^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$.

A consequence of this is the spectral representation of simple max-stable processes.

## Spectral representation of the simple max-stable processes

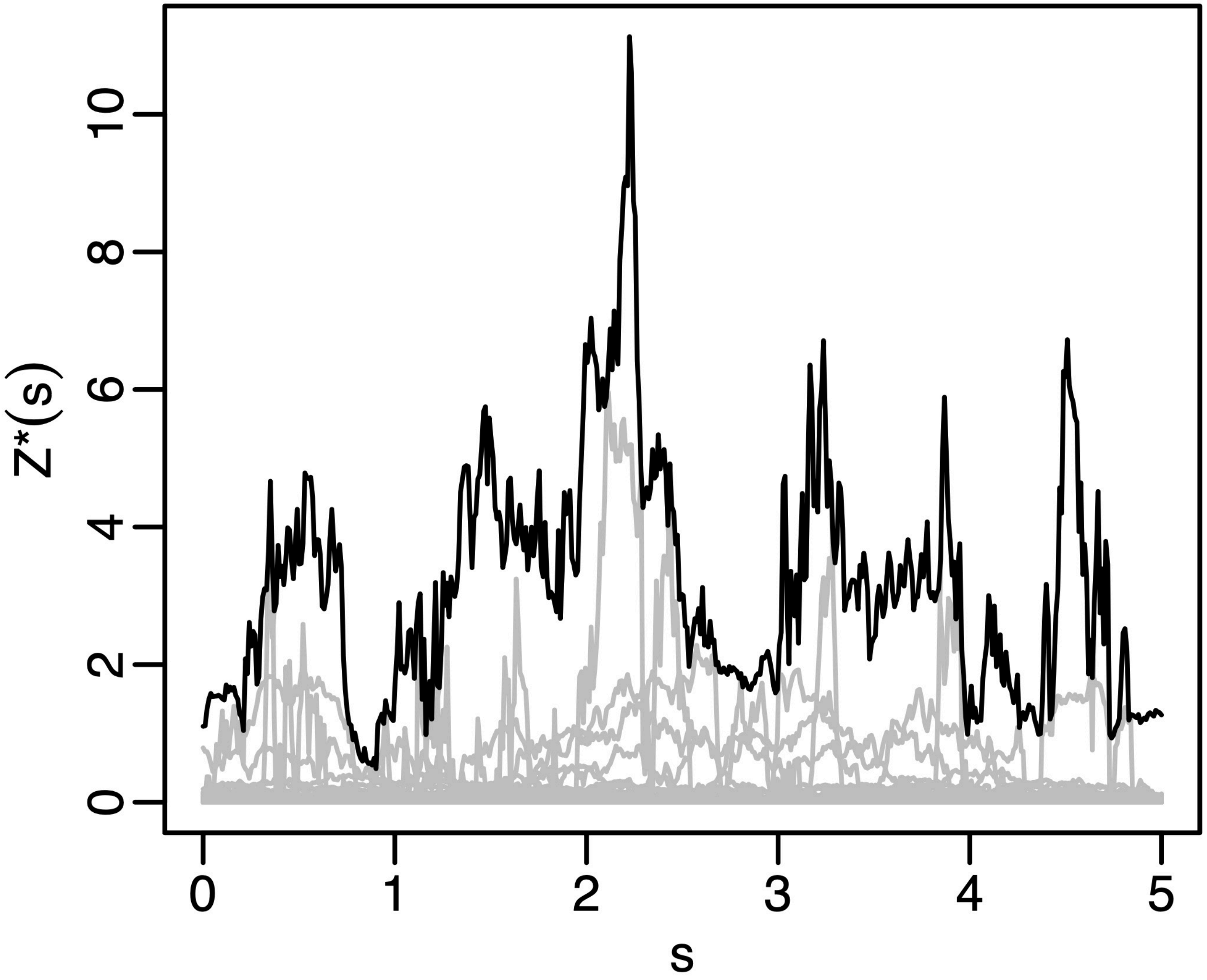
With notations as above, any simple max-stable process $\boldsymbol{Z}^\star$ can be constructed as

$$Z^\star(s) = \max_{i \in \mathbb{N}} R_i W_i(s),$$

and any such construction is a simple max-stable process.

# Illustration: simple max-stable construction

- In gray, “points” $\boldsymbol{P}_i^\star$ of the Poisson process on $D = [0, 5]$
- Max-stable process is the componentwise maximum (in black)



# Simulation based on the spectral representation

If it is simple to simulate from the distribution $F_{\boldsymbol{W}}$ of the spectral process $\boldsymbol{W}$, we can draw samples from the simple max-stable process $\boldsymbol{Z}^\star$.

## Exact simulation

If $P(W_j \leq w_0) = 1$ for some threshold value $0 < w_0 < \infty$, $j = 1, \ldots, d$, then we can perform **exact simulation of $\boldsymbol{Z}^\star$** (even if $Z_j^\star = \max_{i \in \mathbb{N}} R_i W_{ij}$ is defined as a maximum over an infinite number of components):

1. set $m = 1$
2. generate $E_m \sim \mathrm{Exp}(1)$
3. generate $\boldsymbol{W}_m = (W_{m1}, \ldots, W_{md})^T \sim F_{\boldsymbol{W}}$
4. set $\boldsymbol{Z}^\star = (Z_1^\star, \ldots, Z_d^\star)^T$ with $Z_j^\star = \max_{i=1,\ldots,m} \frac{W_{ij}}{\sum_{k=1}^{i} E_k}$ for $j = 1, \ldots, d$
5. IF $\frac{w_0}{\sum_{k=1}^{m} E_k} \leq \min_{j=1,\ldots,d} Z_j^\star$ RETURN $\boldsymbol{Z}^\star$
   ELSE set $m = m + 1$ and GO TO 2

**Remarks:**

- If the distribution of $W_j$ is not finitely bounded, we can fix $w_0$ such that $P(W_j > y_0)$ becomes very small and perform approximation simulation.
- Even with unbounded $W_j$, exact simulation remains possible for many models using different algorithms (see the review of Oesting, Strokorb, 2022).

# Example: Log-Gaussian spectral processes

A possible construction uses a **centered Gaussian process** $\tilde{W}(s)$ with variance function $\sigma^2(s)$ and sets

$$W(s) = \exp(\tilde{W}(s) - \sigma^2(s)/2)$$

$\Rightarrow$ **A class of popular max-stable models:**

- Multivariate: **Huesler–Reiss distributions**
- Spatial: **Brown–Resnick processes**

**Remark:** The distribution of the simple max-stable process $\boldsymbol{Z}^\star = \{Z^\star(s),\, s \in D\}$ depends only on the variogram

$$\gamma(s_1, s_2) = \mathrm{Var}(\tilde{W}(s_2) - \tilde{W}(s_1)), \quad s_1, s_2 \in D.$$

# Illustration: Simulation of Brown–Resnick processes

Two realisation of a spatial Brown-Resnick process
(obtained using the `rmaxstab` function of the `SpatialExtremes` package)
Simulation on a grid $20 \times 20$ (such that $d = 400$) in the square $[0, 10]^2$.

**Illustration:** process $\log(Z^{\star}(s))$ (with standard Gumbel margins)