Uni-variate and Bi-variate Inverted Exponential-Teissier Distribution in Bayesian and Non-Bayesian Framework to Model Stochastic Dynamic Variation of Climate data

Sumangal Bhattacharya
Department of Mathematics and Statistics
Indian Institute of Technology Tirupati, India
sumangal.82@gmail.com

Stochastic Geometry Days 2023

June 14, 2023

Outline

- AIM
- Literature Survey
- Inverted Exponential-Teissier(IET) Distribution
- Statistical Properties
- Characterization of IET
- Measure of Reliability
- Bivariate Inverted Exponential-Teissier(IET) Distribution
- Measure of Dependence
- Survival Function and Hazard Function
- Simulation Study
- Application to Kerala Rainfall Data
- Summary

AIM

- To construct univariate and Bivariate Inverted Exponential-Teissier(IET) Distribution.
- To discuss its statistical properties.
- To illustrate the model using a simulation study.
- To apply it to Kerala rainfall data and compare it with the existing models.

Literature Survey

Uni-variate and Bi-Variate probability distributions are very much indispensable to modeling different types of extreme spatial and spatio-temporal events like environmental, climate, rainfall, drought, and pollution data sets. In the literature, we found the followings:

- (Van Montfort and Witter (1986)) first employ probability distribution to model rainfall depth by Generalized Pareto distribution.
- (Aksoy (2000)) modeled daily rainfall amount and ascension curve with the help of two-parameter Gamma distribution.
- (B. Moccia et al. (2021)) explained the behavior of daily rainfall data in two geographic locations, namely, Lazio and Sicily, located in central and south Italy, are explained using six probability distributions, Frechet, Gumbel, Pareto type-II, Weibull, and Log-normal.
- (G. Teissier (1934)) introduces Teissier Distribution (TD) to the model of the frequency of mortality because of aging.

Literature Survey

- (Hanum et al. (2015)) explained the extreme rainfall event by a new family of probability distribution with the help of T-X transformation.
- (V.K. Sharma et al. (2022)) introduces Exponentiated Teissier Distribution (ETD).
- (N. Poonia et al. (2022)) has introduced Alpha Power Exponentiated Teissier distribution (APETD) and estimated the parameter using MLE, and illustrated the utility of this probability distribution to model the rainfall and temperature data.

Limitations

There are a few limitations:

1. They don't pay enough attention to estimating the parameters in the Bayesian framework, and missing data are inevitable in real-life data sets.
2. They ignore extending those distributions in the Bi-variate or multivariate scenario. Acceptance of the absence of covariates causes significant inaccuracy in prediction.
3. They explain the extreme environmental events disregarding their dependency analysis.

Teissier Distribution

Let X follows the Teissier distribution with parameters $\theta(>0)$.
Then, the cumulative distribution function (CDF) of X is given by:

$$F_X(x) = 1 - \exp\left(\theta x - \exp(\theta x) + 1\right) \qquad x > 0, \theta > 0. \tag{1}$$

In this case, we write $T \sim TD(\theta)$.
The probability density function (PDF) of random variable X is given by:

$$f_X(x) = \theta(\exp(\theta x) - 1)\exp\left(\theta x - \exp\theta x + 1\right) \qquad x > 0, \theta > 0. \tag{2}$$

Inverted Exponential-Teissier Distribution(IET)

A modified Exponential-Teissier distribution function (ET) using T-X transformation is given by:

$$F_{ET}(x) = 1 - \exp\left[\alpha(\theta x - \exp(\theta x) + 1)\right] \qquad x > 0, \theta > 0, \alpha > 0. \tag{3}$$

where T is considered as Exponential distribution with rate parameter $\frac{1}{\alpha}$, and the X is considered as Teissier Distribution with parameter θ.

Let $X \sim ET(\alpha, \theta)$.
Then, $Y = \frac{1}{X} \sim IET(\Theta)$, where $\Theta = \{\alpha, \theta\}$.

Inverted Exponential-Teissier Distribution

The CDF and PDF of IET are given by is given by:

$$
\begin{aligned}
F_{IET}(y;\Theta) &= P(Y \leq y) \\
&= P\left(\frac{1}{X} \leq y\right) = P\left(X \geq \frac{1}{y}\right) = e^{\left(\alpha\cdot\left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)\right)} \\
f_{IET}(y;\Theta) &= e^{\left(\alpha\cdot\left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)\right)} \cdot \frac{\alpha\theta\left(e^{\frac{\theta}{y}} - 1\right)}{y^2}; \\
&\quad where, y > 0,\ \Theta = \{(\alpha,\theta) : \alpha > 0; \theta > 0\}
\end{aligned} \tag{4}
$$

CDF and PDF of IET

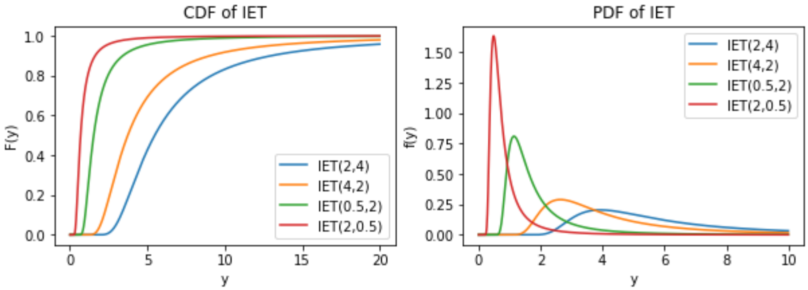

Figure: The characteristics of CDF (*left*) and PDF (*right*) of $IET(\alpha, \theta)$ for different values of (α, θ).

Expection E(Y)

Let $Y \sim IET(\Theta)$.

$$
\begin{aligned}
E[Y] &= \int_0^\infty 1 - e^{\alpha\cdot\left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)} dy \\
&= \int_0^\infty 1 - \sum_{r=0}^{\infty} \frac{\left(\alpha\cdot(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1)\right)^r}{r!} dy \\
&= -\int_0^\infty \sum_{r=1}^{\infty} \frac{\left(\alpha\cdot(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1)\right)^r}{r!} dy \\
&= \sum_{r=1}^{\infty}\sum_{k=0}^{r} \frac{(-1)^{k+1}\alpha^r\binom{r}{k}}{r!} \cdot \int_0^\infty \left(e^{\frac{\theta}{y}} - \frac{\theta}{y}\right)^k dy
\end{aligned}
\tag{5}
$$

Mode (M_0) and Quantile [$Q(p)$] of IET

- Mode(M_0) of Y is given by

$$e^{\frac{\theta}{M_0}} + e^{-\frac{\theta}{M_0}} = \frac{1}{\alpha} + 2$$
$$\implies \cosh\left(\frac{\theta}{M_0}\right) = 1 + \frac{1}{2\alpha} \tag{6}$$
$$\implies M_0 = \frac{\theta}{\cosh^{-1}\left(\frac{2\alpha+1}{2\alpha}\right)}$$

- p^{th} Quantile [$Q(p)$] of Y can be calculated from the following equality

$$\frac{\theta}{Q(p)} - e^{\frac{\theta}{Q(p)}} = \left(\frac{1}{\alpha} \cdot log(p) - 1\right) \tag{7}$$

MGF of IET

Theorem

$Y \sim IET(\Theta)$ *iff*

$$(i)\ M_Y(t) = \alpha\theta \sum_{p=2}^{\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{p} \frac{(t\alpha)^p \cdot (-1)^j \cdot \binom{p}{j} \cdot \theta^k}{p!k!} \int_0^{\infty} \left(e^{\frac{\theta}{y}} - \frac{\theta}{y}\right)^j \cdot y^{p-2-k} dy$$

$$(ii)\ \phi_Y(t) = \alpha\theta \sum_{p=2}^{\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{p} \frac{(it\alpha)^p \cdot (-1)^j \cdot \binom{p}{j} \cdot \theta^k}{p!k!} \int_0^{\infty} \left(e^{\frac{\theta}{y}} - \frac{\theta}{y}\right)^j \cdot y^{p-2-k} dy$$

$$(iii)\ \phi_Y(t) = \frac{M_Y(t)}{(i-1)},\ \text{where } i = \sqrt{-1}.$$

Survival and Hazard Function of IET

The probability of lifetime (Y) of a component surviving beyond time y is survival function $[S_Y(y)]$ and the corresponding conditional failure rate, i.e., hazard rate $[h_Y(y)]$ are given in the followings:

$$S_Y(y;\Theta) = P(Y \geq y) = 1 - e^{\left[\alpha\cdot\left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)\right]}$$

$$h_Y(y;\Theta) = \frac{e^{\left[\alpha\cdot\left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)\right]} \cdot \alpha\theta\left(e^{\frac{\theta}{y}} - 1\right)}{y^2 \cdot \left[1 - \exp\left(\alpha\cdot\left(\frac{\theta}{y} - e^{\frac{\theta}{y}} + 1\right)\right)\right]} \quad (8)$$

$$where,\ y > 0; \& \, \Theta = \{(\alpha,\theta) : \alpha > 0; \theta > 0\}$$

Behaviour of Hazard Rate of IET

Theorem

If $Y \sim IET(\Theta)$ then $h_Y(y;\Theta)$ is
(i) Increasing if
$$\left((2y+\theta)\cdot e^{\frac{\theta}{y}}-2y\right)\cdot e^{\alpha\cdot\left(\frac{\theta}{y}-e^{\frac{\theta}{y}}+1\right)}+\alpha\theta e^{\frac{2\theta}{y}} \geq \left((2\alpha+1)\theta+2y\right)\cdot e^{\frac{\theta}{y}}-2y-\alpha\theta.$$
(ii) Decreasing if
$$\left((2y+\theta)\cdot e^{\frac{\theta}{y}}-2y\right)\cdot e^{\alpha\cdot\left(\frac{\theta}{y}-e^{\frac{\theta}{y}}+1\right)}+\alpha\theta e^{\frac{2\theta}{y}} \leq \left((2\alpha+1)\theta+2y\right)\cdot e^{\frac{\theta}{y}}-2y-\alpha\theta.$$

Survival Function (SF) and Hazard Rate (HR) of IET

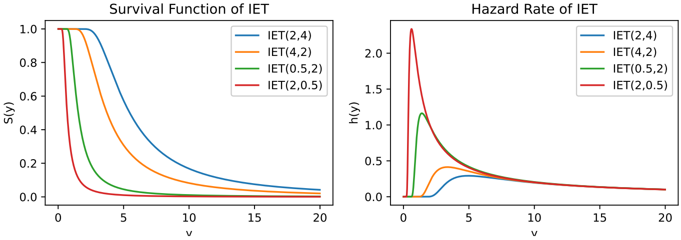

Figure: The characteristics of SF (*left*) and HR (*right*) of $IET(\Theta)$ for different values of Θ

.

Multi-Dimensional Copulas

Definition (d- Dimensional Copulas)

A function $C : [0,1]^d \to [0,1]$ is a d-dimensional copula if the following conditions are satisfied:

1. $C(1,\ldots,y_j,\ldots,1) = y_j, \forall\ j = 1,2,\ldots,d$ with $y_j \in [0,1]$
2. $C(y_1,y_2,\ldots,y_d) = 0$ if at least one $y_j = 0$ for $j = 1,2,\ldots,d$
3. For any $y_{\{j,1\}}, y_{\{j,2\}} \in [0,1]$ with $y_{\{j,1\}} \leq y_{\{j,2\}}$, for $j = 1,2,\ldots,d$:

$$\sum_{a_1=1}^{2}\sum_{a_2=1}^{2}\cdots\sum_{a_d=1}^{2}(-1)^{a_1+a_2+\ldots+a_d}C(y_{\{1,a_1\}},y_{\{2,a_2\}},\ldots,y_{\{d,a_d\}}) \geq 0 \qquad (9)$$

Multivariate Gaussian Copula

Example

The multivariate Gaussian copula [R. B. Nelsen (2007)] is given by the function:

$$C(u_1, u_2, \ldots, u_d | R) = \mathbf{\Phi}_R(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_d)), \tag{10}$$

where Φ^{-1} is the inverse of the cumulative distribution function of the standard normal distribution and $\mathbf{\Phi}_R$ is the joint CDF of a standard multivariate normal distribution with correlation matrix R.

FGM Copula

Example

The Farlie-Gumbel-Morgenstern (FGM) copula is given by

$$C(x,y) = xy(1+\rho(1-x)(1-y)) \tag{11}$$

where $(x,y) \in [0,1]\times[0,1]$, $-1 \le \rho \le 1$.

Sklar's Theorem

Theorem (Sklar (1959))

Sklar's Theorem: *Let $Y_1, Y_2, \ldots, Y_d$ be random variables with marginal distribution functions $F_1, F_2, \ldots, F_d$ and joint cumulative distribution function F, then the following are true:*

1. *There exists a d-dimensional copula C such that for all $y_1, y_2, \ldots, y_d \in \mathbb{R}$*

$$F(y_1, y_2, \ldots, y_d) = C(F_1(y_1), F_2(y_2), \ldots, F_d(y_d)) \tag{12}$$

2. *If $Y_1, Y_2, \ldots, Y_d$ are continuous then the copula C is unique. Otherwise, C can be uniquely determined on a d-dimensional rectangle $Range(F_1) \times Range(F_2) \times \ldots \times Range(F_d)$.*

Joint CDF and PDF of Bi-variate IET (BIET)

Let $X \sim IET(\Theta_1)$ and $Y \sim IET(\Theta_2)$. The joint CDF using copula C is given by

$$H(x,y) = C(F_X(x)), F_Y(y))$$

. In particular, if we consider FGM copula, the CDF, PDF and Conditional PDF are given, respectively

$$\begin{aligned} H(x,y \mid \Theta_1,\Theta_2,k) &= F(x)\cdot F(y)\cdot(1+k\cdot(1-F(x))\cdot(1-F(y))) \\ &= e^{\left(\alpha_1\cdot\left(\frac{\theta_1}{x}-e^{\frac{\theta_1}{x}}+1\right)\right)}\cdot e^{\left(\alpha_2\cdot\left(\frac{\theta_2}{y}-e^{\frac{\theta_2}{y}}+1\right)\right)}\cdot \end{aligned}$$

$$\left(1+k\cdot\left(1-e^{\left(\alpha_1\cdot\left(\frac{\theta_1}{x}-e^{\frac{\theta_1}{x}}+1\right)\right)}\right)\cdot\left(1-e^{\left(\alpha_2\cdot\left(\frac{\theta_2}{y}-e^{\frac{\theta_2}{y}}+1\right)\right)}\right)\right)$$

$$\begin{aligned} g(x,y \mid \Theta_1,\Theta_2,k) &= f(x)\cdot f(y)\cdot(1+k\cdot(1-2\cdot F(x))\cdot(1-2\cdot F(y))) \\ g(x \mid y,\Theta_1,\Theta_2,k) &= f(x)\cdot(1+k\cdot(1-2\cdot F(x))\cdot(1-2\cdot F(y))) \end{aligned} \tag{13}$$

Behaviour of PDF and CDF of BIET

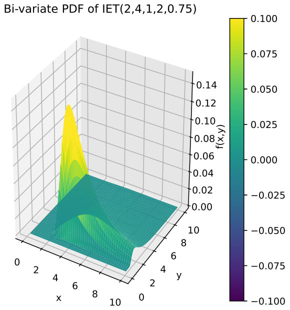

Figure: *(left)* The characteristic of PDF of $BIET(2,4,1,2,0.75)$ and in *(right)* that of CDF of $BIET(2,4,1,2,0.75)$.

Behaviour of SF and HR of BIET

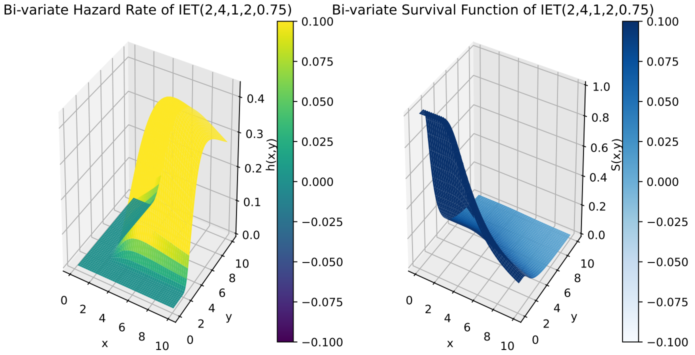

Figure: *(left)* The characteristic of HR of $BIET(2,4,1,2,0.75)$ and in *(right)* that of SF of $BIET(2,4,1,2,0.75)$.

Dependency Measurements

- **Kendall's τ:**

$$\begin{aligned}\tau &= P\left[(X_1 - X_2)\cdot(Y_1 - Y_2) > 0\right] - P\left[(X_1 - X_2)\cdot(Y_1 - Y_2) < 0\right] \\ &= 1 - 4\cdot\int_0^1\int_0^1 \frac{\partial H(x,y \mid \Theta_1,\Theta_2,k)}{\partial F(x)}\cdot\frac{\partial H(x,y \mid \Theta_1,\Theta_2,k)}{\partial F(y)}\, dF(x)dF(y) \\ &= 1 - \frac{(9-2k)}{9} = \frac{2\cdot k}{9}\end{aligned} \tag{14}$$

- **Speraman's ρ:**

$$\begin{aligned}\rho_s &= 12k\cdot\int_0^1\int_0^1 kF(x)\cdot F(y)\,(1-F(x))\cdot(1-F(y))\,dF(x)dF(y) \\ &= 12k\cdot\int_0^1 F(x)\cdot(1-F(x))\,dF(x)\int_0^1 F(y)\cdot(1-F(y))\,dF(y) \\ &= \frac{12k}{36} = \frac{k}{3}\end{aligned} \tag{15}$$

Dependency Measurement

- **Schweizer and Wolff's σ:**

$$\begin{aligned}\sigma_{X,Y} &= 12k\cdot\int_0^1 F(x)\cdot(1-F(x))\,dF(x)\int_0^1 F(y)\cdot(1-F(y))\,dF(y)\\ &= \frac{12k}{36} = \frac{k}{3}\end{aligned} \tag{16}$$

- **Local Dependence $\gamma(x,y)$:**

$$\frac{\partial^2 log(f(x,y\mid\Theta,k))}{\partial x\partial y} = 4\cdot\left[\frac{f(x)f(y)m_2(x,y)-2kf(x)m_1(x,y)(1-2F(y))}{\Big((1+k(1-2F(x))\cdot(1-2F(y))\Big)^2}\right] \tag{17}$$

$m_1(x,y)=f(y)(1-2F(x))$; $m_2(x,y)=1+k(1-2F(x))\cdot(1-2F(y))$.

Tail behaviour and Tail dependency

- Here $H(x,y) \geq F(x) \cdot F(y) \; \forall \; k > 0$ therefore, X and Y are Positive Quadrant Dependent (PQD). Moreover, X and Y are Negative Quadrant Dependent (NQD) $\forall \; k < 0$.

Theorem

Let $(X, Y) \sim BIET(\Theta_1, \Theta_2, k)$ if $k \geq 0$ then, Y is Left-Tail Decreasing (LTD), Right-Tail Increasing (RTI), and Stochastically Increasing (SI) on X.

Probabilistic Time-Series Forecasting

We assume that the joint Stochastic Process (SP), $\left(Y(t), Y(t+\tau)\right) \sim BIET(\Theta_1, \Theta_2, k)$ *where* $t \in T$, and τ is a temporal lag. In this section, we introduce a novel method of probabilistic time series prediction outlined by $BIET(\Theta_1, \Theta_2, k_\tau)$.

$$
\begin{aligned}
& P\left[Y(t+\tau) \leq y_{t+\tau} | Y(t) = y_t\right] = \frac{1}{2} \\
\Rightarrow & P\left[F(Y(t+\tau)) \leq F(y_{t+\tau}) | F(Y(t)) = F(y_t)\right] = \frac{1}{2} \\
\Rightarrow & \frac{\partial H(y_t, y_{t+\tau})}{\partial F(Y_t)} = \frac{1}{2} \\
\Rightarrow & 2F(y_t) + 2k_\tau F(y_t) - 2k_\tau F^2(y_t) = F(y_{t+\tau})\left(2k_\tau F(y_t) - 2k_\tau F^2(y_t)\right) \\
& \Rightarrow F(y_{t+\tau}) = \frac{F(y_t)\cdot\left(1 + k_\tau(1 - F(y_t))\right) - 1}{k_\tau \cdot F(y_t)\cdot(1 - F(y_t))} \\
& \Rightarrow y_{t+\tau} = F^{-1}\left[\frac{F(y_t)\cdot\left(1 + k_\tau(1 - F(y_t))\right) - 1}{k_\tau \cdot F(y_t)\cdot(1 - F(y_t))}\right]
\end{aligned}
\tag{18}
$$

Simulate Data from IET Distribution

Algorithm 2 Simulated data set generated from Uni-variate $IET(\Theta)$

```
    Input n                                   ▷ n = Sample Size
          m                                   ▷ m = Number of Iterations
    Output S ∈ R^{m×n}
  for each i ← 1 to m do
      for each j ← 1 to n do
          Generate u_ij ~ Uniform(0, 1)
          S[i, j] ← F^{-1}(u_ij)    ▷ where F^{-1} is the quasi-inverse of the IET CDF.
      end for
  end for
```

Simulate Data from BIET Distribution

Algorithm 3 Simulated data set generated from $BIET(\Theta_1, \Theta_2, k)$

Input n ▷ n = Sample Size

m ▷ m = Number of Iterations

Output $S_X, S_Y \in \mathbf{R}^{m \times n}$

for each i ← 1 to m **do**

  for each j ← 1 to n **do**

    Generate $(u_{ij}, t_{ij}) \sim Uniform\left[(0,1) \times (0,1)\right]$

    $v_{ij} = C_u^{-1}(t_{ij})$ ▷ $C_u = \frac{\partial C(u,v)}{\partial u}$

    $S_X[i,j] \leftarrow F_1^{-1}(u_{ij})$

    $S_Y[i,j] \leftarrow F_2^{-1}(v_{ij})$

  end for

end for

MLE

Table: MLE of Θ in $IET(\Theta)$

n	Parameter	Sample Mean	MSE	Bias	BCI(0.95)	BSE
100	$\alpha = 2$	1.470	1.588	-0.529	(0.351, 4.723)	1.143
	$\theta = 3$	3.843	1.763	0.843	(2.057, 5.748)	1.025
200	$\alpha = 2$	1.500	1.226	-0.499	(0.419, 4.337)	0.988
	$\theta = 3$	3.670	1.181	0.676	(2.166, 5.448)	0.851
300	$\alpha = 2$	1.498	1.335	-0.501	(0.471, 4.358)	1.041
	$\theta = 3$	3.646	1.021	0.648	(2.122, 5.191)	0.777

Bayesian Estimation

Table: BE of Θ in $IET(\Theta)$

n	Parameter	Sample Mean	MSE	Bias	BCI(0.95)	BSE
100	$\alpha = 1$	0.755	0.151	-0.244	(0.202, 1.294)	0.304
	$\theta = 3.5$	4.132	0.909	0.632	(3.220,5.785)	0.713
200	$\alpha = 1$	0.729	0.167	-0.270	(0.223, 1.323)	0.167
	$\theta = 3.5$	4.166	0.974	0.666	(3.198, 5.801)	0.974
300	$\alpha = 1$	0.707	0.181	-0.292	(0.224, 1.291)	0.309
	$\theta = 3.5$	4.221	1.107	0.721	(3.206,5.868)	0.766

Weighted Least Square Estimation

Table: WLS of Θ in $IET(\Theta)$

n	Parameter	Sample Mean	MSE	Bias	BCI(0.95)	BSE
100	$\alpha = 1$	1.219	1.425	0.219	(0.171, 4.595)	1.173
	$\theta = 3$	3.357	1.335	0.357	(1.567,5.552)	1.099
200	$\alpha = 1$	1.091	0.896	0.091	(0.225, 3.863)	0.942
	$\theta = 3$	3.308	0.892	0.308	(1.698, 5.204)	0..893
300	$\alpha = 1$	1.123	0.923	0.123	(0.241, 3.937)	0.953
	$\theta = 3$	3.267	0.838	0.266	(1.690, 5.058)	0.876

Cramer Von-Mises Estimation

Table: CVM estimate of Θ in $IET(\Theta)$

n	Parameter	Sample Mean	MSE	Bias	BCI(0.95)	BSE
100	$\alpha = 1$	1.067	1.128	0.066	(0.170, 4.800)	1.060
	$\theta = 3$	3.516	1.450	0.516	(1.608, 5.619)	1.088
200	$\alpha = 1$	1.064	0.886	0.064	(0.211, 3.919)	0.939
	$\theta = 3$	3.364	0.966	0.364	(1.691, 5.287)	0.913
300	$\alpha = 1$	1.004	0.588	0.004	(0.255, 3.149)	0.767
	$\theta = 3$	3.326	0.742	0.326	(1.870, 4.924)	0.797

EM Estimation

Table: EM estimate of Θ in $IET(\Theta)$

n	Parameter	Sample Mean	MSE	Bias	BCI(0.95)	BSE
100	$\alpha = 2$	1.461	2.013	-0.538	(0.280, 5.146)	1.313
	$\theta = 3$	3.991	2.315	0.991	(1.981, 5.909)	1.154
200	$\alpha = 2$	1.408	1.956	-0.591	(0.308, 5.100)	1.267
	$\theta = 3$	4.028	2.357	1.028	(1.988, 5.877)	1.140
300	$\alpha = 2$	1.472	1.762	-0.527	(0.342, 5.021)	1.218
	$\theta = 3$	3.896	1.954	0.896	(2.024, 5.786)	1.073

Simulation Study of BIETD

Table 9: MLE estimate of Θ_1, and Θ_2 in $BIET(\Theta_1, \Theta_2, k)$

n	Parameter	True Value	MLE	MSE	Bias	BSE	CI(0.95)
100	α_1	2	1.91894	1.318571	-0.08105857	1.17219	(0.713956, 5.048073)
	θ_1	4	4.3918	1.156687	0.3918002	0.9293735	(2.6501084, 5.905896)
	α_2	1	1.18307	0.999449	0.1830769	1.01724	(0.18919609, 4.393151)
	θ_2	2	2.19754	0.4182949	0.1975495	0.6473151	(1.09674574, 3.567457)
	k	0.5	0.542354	0.07106858	0.04235385	0.25683	(0.06647816, 0.99987)
200	α_1	2	1.810679	0.9559668	-0.1893206	1.04497	(0.7186286, 4.7243299)
	θ_1	4	4.442419	0.932662	0.4424185	0.8734426	(2.779292, 5.89826)
	α_2	1	1.111495	0.3376843	0.1114951	0.7936206	0.2978534, 3.4017358)
	θ_2	2	2.110467	0.2221527	0.1104672	0.4794214	(1.2047094, 3.1016042)
	k	0.5	0.5030928	0.04734568	0.003092806	0.204922	(0.1140134, 0.9121645)
300	α_1	2	1.69689	0.715872	-0.3031097	0.9249394	(0.7393568, 4.3671022)
	θ_1	4	4.511125	0.7528599	-0.5111251	0.8093461	(2.8509058, 5.8702773)
	α_2	1	1.060169	0.3384617	0.0601688	0.628002	(0.3535171, 2.7387518)
	θ_2	2	2.087978	0.150146	0.087978	0.3985438	(1.3251008, 2.8927158)
	k	0.5	0.5004218	0.02269951	0.00042178	0.1655991	(0.178608, 0.8276942)

Kerala Rainfall Data

Table: Fitted PDF on Summer Rainfall Data

PDF	MLE	-logL	AIC	BIC
IET	$\hat{\alpha} = 14.54, \hat{\theta} = 45.23$	664.5	1333.006	1338.53
ETD	$\hat{\alpha} = 3.66, \hat{\theta} = 0.00343$	666.844	1337.689	1343.213
APETD	$\hat{\alpha} = 6.95, \hat{\theta} = 0.00544, \hat{\gamma} = 0.367$	677.322	1358.644	1368.93
Weibull	$\hat{\alpha} = 2.621, \hat{\theta} = 152.744$	973.4278	1950.856	1951.618
Gamma	$\hat{\alpha} = 86.47, \hat{\theta} = 0.76$	2047.882	4099.764	4100.526

Kerala Rainfall Data

Table: Fitted PDF on Monsoon Rainfall Data

PDF	MLE	-logL	AIC	BIC
IETD	$\hat{\alpha} = 0.01137, \hat{\theta} = 2360$	736.852	1477.705	1483.229
ETD	$\hat{\alpha} = 0.5084, \hat{\theta} = 0.0024773$	765.3679	1534.736	1540.26
APETD	$\hat{\alpha} = 2.459, \hat{\theta} = 0.0026, \hat{\gamma} = 4.216$	724.7513	1453.503	1463.789
Weibull	$\hat{\alpha} = 6.326, \hat{\theta} = 590.17$	1153.67	2311.34	2312.102
Gamma	$\hat{\alpha} = 579.3027, \hat{\theta} = 1.2$	1783.327	3570.654	3571.416

Kerala Rainfall Data

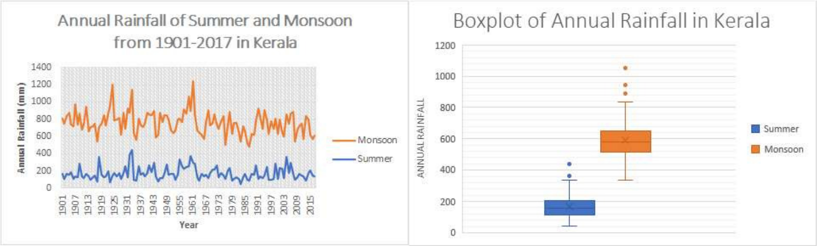

Figure: Behaviour of Kerala rainfall in summer and winter

Fitting suitable PDFs

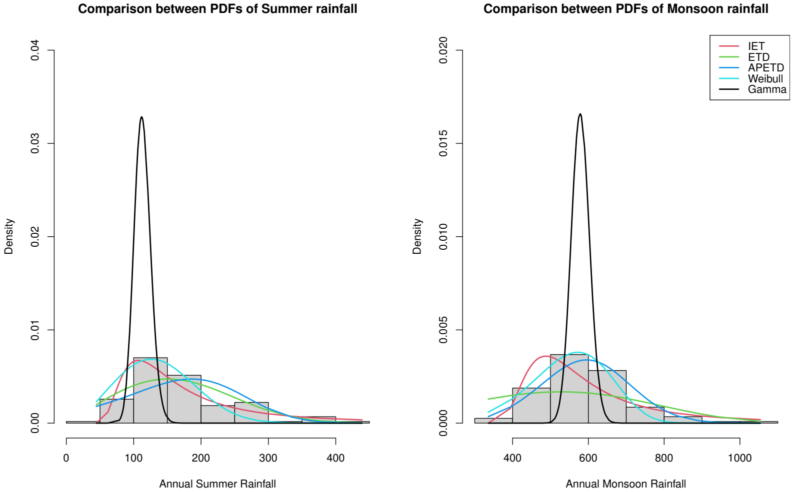

Figure: Comparison between the Five PDFs in Rainfall Data in Kerala.

Copula-Based Median Regression in Kerala Rainfall Data

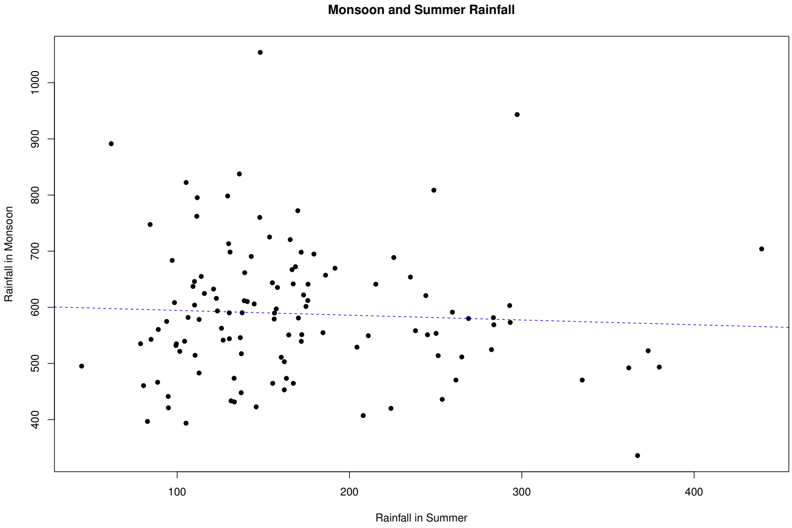

Figure: BIET-based Median regression between annual Summer and Annual Monsoon rainfall data in Kerala.

Probabilistic Forecasting

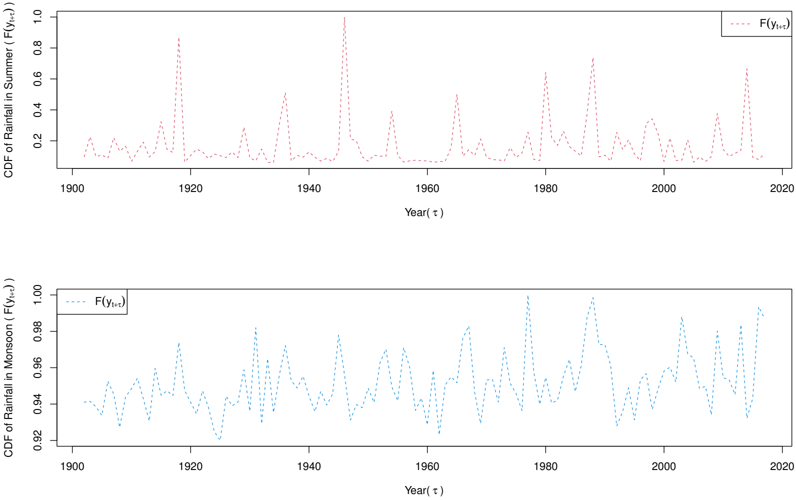

Figure: Temporal behavior of CDF of summer and monsoon rainfall.

Summary

- We introduced a new IET distribution, derived its different statistical properties and characterizations, and measured various reliability properties and also estimate model parameters using techniques such as MLE, EM, WLSE, CVM, BE, etc.
- The proposed IET distribution shows better modeling compatibility than existing distributions in the literature.
- We extended IET to BIET and explored different measures of dependency and also introduced a new temporal probabilistic median regression model to explain the probability that a variable is less than some particular value.
- we illustrate our entire derivations using Kerala rainfall data of summer and monsoon from 1901 to 2017 as a case study.
- The proposed model is applied to real data and compared with existing models in the literature.

References

- **Moccia, B., Mineo, C., Ridolfi, E., Russo, F., Napolitano, F.**: Probability distributions of daily rainfall extremes in Lazio and sicily, Italy, and design rainfall inferences. *Journal of Hydrology: Regional Studies 33, 100771(2021).*
- **Teissier, G.**: *Recherches sur le vieillissement et sur les lois de la mortalit ´e. ii. essai d'interpr´etation gen´erale des courbes de survie.* Annales de physiologie et de physicochimie biologique 10, 260–284 (1934).
- **Sharma, V.K., Singh, S.V., Shekhawat, K.**: Exponentiated Tessier distribution with increasing, decreasing, and bathtub hazard functions. *Journal of Applied Statistics 49(2), 371–393 (2022)*
- **Poonia, N., Azad, S.**: *Alpha power exponentiated Teissier distribution with application to climate datasets.* Theoretical and Applied Climatology, 1–15 (2022)
- **Alzaatreh, A., Lee, C., Famoye, F.**: A new method for generating families of continuous distributions. *Metron 71(1), 63–79 (2013)*
- **R. B. Nelsen**, *An introduction to copulas.* Springer Science and Business Media, 2007.

Thank You