Maximum composite likelihood estimators for Brown-Resnick random fields in infill asymptotics

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I. Introduction

Max-stable distributions

Definition

A random variable Z has a **max-stable distribution** if for any i.i.d. copies $Z_1, \dots, Z_n$ of Z there exist $a_n > 0$, $b_n \in \mathbf{R}$ such that

$$\frac{\bigvee_{i=1}^{n} Z_i - b_n}{a_n} \stackrel{\mathscr{D}}{=} Z.$$

Examples:

1. Fréchet ($\alpha > 0$):

$$F(z) = \begin{cases} 0 & \text{if } z \leq 0, \\ e^{-z^{-\alpha}} & \text{if } z > 0; \end{cases}$$

2. Gumbel:

$$F(z) = e^{-e^{-z}}, \; z \in \mathbf{R};$$

3. Weibull ($\alpha > 0$):

$$F(z) = \begin{cases} e^{-(-z)^{\alpha}} & \text{if } z < 0, \\ 1 & \text{if } z \geq 0. \end{cases}$$

Max-stable random fields

Definition

A stationary random field $\eta = (\eta(x))_{x \in \mathbf{R}^d}$ is **max-stable** if for any i.i.d. copies $\eta_1, \ldots, \eta_n$ of η there exist continuous functions $(a_n(x))_{x \in \mathbf{R}^d}$, $(b_n(x))_{x \in \mathbf{R}^d}$, with $a_n(x) > 0$, such that

$$\left(\frac{\bigvee_{i=1}^n \eta_i(x) - b_n(x))}{a_n(x)} \right)_{x \in \mathbf{R}^d} \overset{\mathscr{D}}{=} \eta.$$

- η is **simple** if $\mathbb{P}(\eta(x) \leq z) = e^{-z^{-1}}$, $z > 0$;
- if so, $n^{-1} \bigvee_{i=1}^n \eta_i \overset{\mathscr{D}}{=} \eta$.

Spectral representation

Theorem (de Haan (1984))

Let η be a simple max-stable random field. Then

$$\eta(x) = \bigvee_{i \geq 1} U_i Y_i(x),\ x \in \mathbf{R}^d,$$

where

- (U_i) *is a P.P.P. on* $\mathbf{R}_+$ *with intensity* $u^{-2}\mathrm{d}u$ *and* $U_i > U_{i+1}$*;*
- (Y_i) *are i.i.d. copies of* $Y \geq 0$ *such that* $\mathbb{E}[Y(x)] = 1$, $x \in \mathbf{R}^d$*;*
- (U_i) *and* (Y_i) *are independent.*

Brown-Resnick random fields

Definition

Let

- $\alpha \in (0,2)$, $\sigma^2 > 0$;
- (U_i): P.P.P. on $\mathbf{R}_+$ with intensity $u^{-2}\mathrm{d}u$ and $U_i > U_{i+1}$;
- (W_i): i.i.d. fractional Brownian random fields, i.e. Gaussian random fields with stationary increments, $W(0) = 0$ and $\mathbb{V}[W(x)] = \sigma^2 \|x\|^\alpha$.

The random field η is called **Brown-Resnick** if

$$\eta(x) = \bigvee_{i \geq 1} U_i \exp\left(W_i(x) - \frac{1}{2}\sigma^2 \|x\|^\alpha\right),\ x \in \mathbf{R}^d.$$

Simulation

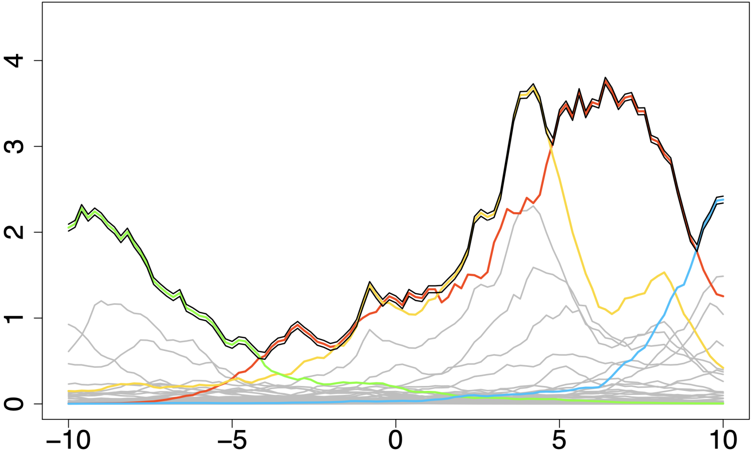


Figure: Brown-Resnick for $d = 1$ (Dombry, Engelke & Oesting)

Known results for Brown-Resnick

- modelization for spatial rainfall [de Haan, Zhou; 08];
- stationarity, isotropy, mixing [Kabluchko, Schlather, de Haan; 09];
- computations of laws and conditional laws [Dombry, Eyi-Minko; 13]
- inference for independent replicates [Hüser, Davison; 13]
- simulations [Dieker, Mikosch; 15];
- canonical tessellation [Dombry, Kabluchko; 18]
- biased CLT for power variations [Robert; 20]

Canonical tessellation

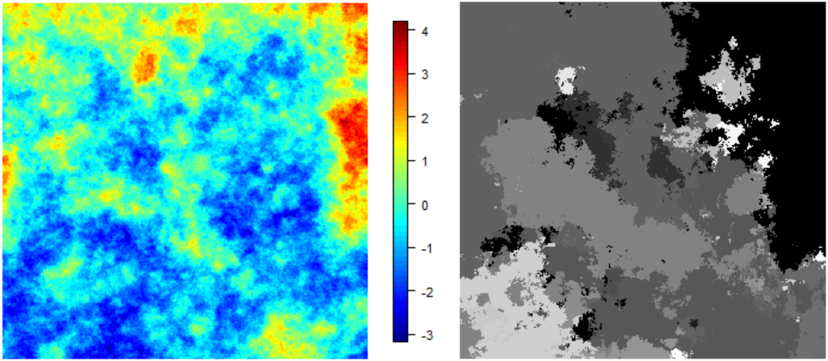


Figure: Brown-Resnick and its canonical tessellation (Dombry & Kabluchko)

Infill asymptotics

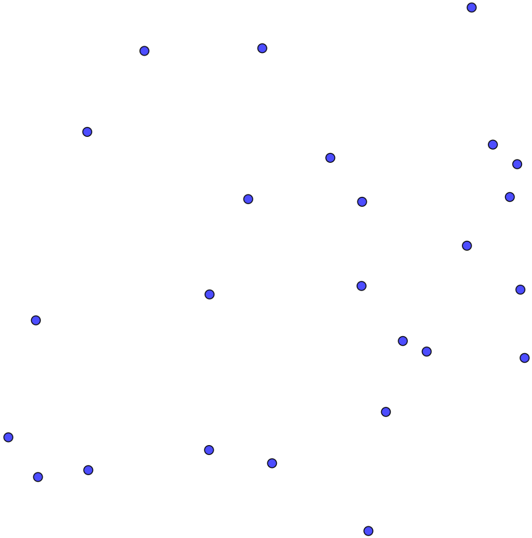
Question: $\widehat{\sigma}$, $\widehat{\alpha}$ when the data are observed in a **fixed window**?

Main difficulties:

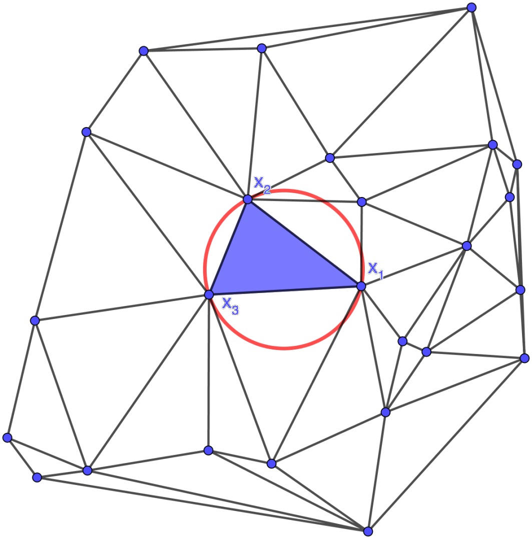
- non-Gaussian random field;
- no explicit formula for finite-dimensional distributions;
- no explicit representation for maximum likelihood estimators;
- no mixing properties.

II. Main result

Poisson point process



Delaunay triangulation



Composite likelihood estimators

Delaunay triangulation:

- P_N: P.P.P. of intensity N in $\mathbf{R}^2$ (sampling scheme);
- E_N: set of edges (x_1, x_2) with $x_1 \preceq x_2$ and $x_1 \in [0,1]^2$;
- DT_N: set of triangles (x_1, x_2, x_3), with $x_1 \preceq x_2 \preceq x_3$ and $x_1 \in [0,1]^2$.

Objective functions:

- pairwise composite log-likelihood function:

$$\ell_{2,N}(\sigma, \alpha) = \sum_{(x_1, x_2) \in E_N} \log f_{x_1, x_2}(\eta(x_1), \eta(x_2));$$

- triplewise composite log-likelihood function:

$$\ell_{3,N}(\sigma, \alpha) = \sum_{(x_1, x_2, x_3) \in DT_N} \log f_{x_1, x_2, x_3}(\eta(x_1), \eta(x_2), \eta(x_3)).$$

Main theorem (first statement)

Notation: $\widehat{\sigma}^2_{2,N}$, $\widehat{\alpha}_{2,N}$ (pairwise MCLEs).

Theorem

Let $\sigma^2 > 0$ and $\alpha \in (0,1)$. Then there exist $L_{Z_{k\backslash j}}(0) \geq 0$, $k > j$, such that

1. $$N^{\alpha/4}\left(\widehat{\sigma}^2_{2,N} - \sigma^2\right) \xrightarrow{\mathbb{P}} c\sigma^2 \sum_{j\geq 1}\sum_{k>j} L_{Z_{k\backslash j}}(0);$$

2. $$N^{\alpha/4}\log(N)\left(\widehat{\alpha}_{2,N} - \alpha\right) \xrightarrow{\mathbb{P}} c \sum_{j\geq 1}\sum_{k>j} L_{Z_{k\backslash j}}(0).$$

NB: similar results for $\widehat{\sigma}^2_{3,N}$ and $\widehat{\alpha}_{3,N}$.

Tessellation based on two maximizing trajectories

- **Brown-Resnick:**

$$\eta(x) = \bigvee_{i \geq 1} \exp\left(Z_i(x) - \tfrac{1}{2}\sigma^2 \|x\|^\alpha\right)$$

with

$$Z_i(x) = \log U_i + W_i(x), \quad x \in \mathbf{R}^2.$$

- **Tessellation** $(\mathbf{C}_{k,j})_{j<k}$ of $[0,1]^2$ with

$$\mathbf{C}_{k,j} = \left\{x \in [0,1]^2 : Z_k(x) \bigwedge Z_j(x) \geq \bigvee_{i \neq j,k} Z_i(x)\right\}.$$

Local time

- **Occupation measure:** for any Borel subset $B \subset \mathbf{R}$,

$$\nu^{(k\backslash j)}(B) = \int_{\mathbf{C}_{k,j}} \mathbb{1}_{Z_{k\backslash j}(x) \in B} \, \mathrm{d}x,$$

where $Z_{k\backslash j}(x) = Z_k(x) - Z_j(x)$.

- **Local time at level ℓ:**

$$L_{Z_{k\backslash j}}(\ell) := \frac{\mathrm{d}\nu^{(k\backslash j)}}{\mathrm{d}r}(\ell).$$

- **Other expression (limit in L^2):**

$$L_{Z_{k\backslash j}}(\ell) = \frac{1}{2\pi} \lim_{M\to\infty} \int_{-[M,M]} \int_{\mathbf{R}} e^{\mathrm{i}\xi(Z_{k\backslash j}(x)-\ell)} \mathrm{d}x \mathrm{d}\xi.$$

Main theorem (second statement)

Theorem

Let $\sigma^2 > 0$, $\alpha \in (0,1)$ *and let* $L_{Z_{k\setminus j}}(0)$ *be the local time of* $Z_{k\setminus j} = Z_k - Z_j$ *at level 0, with*

$$Z_i(x) = \log U_i + W_i(x), \quad x \in \mathbf{R}^2.$$

Then

1. $$N^{\alpha/4}\left(\widehat{\sigma}^2_{2,N} - \sigma^2\right) \xrightarrow{\mathbb{P}} c\sigma^2 \sum_{j\geq 1}\sum_{k>j} L_{Z_{k\setminus j}}(0);$$

2. $$N^{\alpha/4}\log(N)\left(\widehat{\alpha}_{2,N} - \alpha\right) \xrightarrow{\mathbb{P}} c \sum_{j\geq 1}\sum_{k>j} L_{Z_{k\setminus j}}(0).$$

III. Sketch of proof for $\widehat{\sigma}^2_{2,N}$

MCLE and increments

The pairwise MCLE $\widehat{\sigma}^2_{2,N}$ is such that

$$\frac{\partial}{\partial\sigma}\ell_{2,N}(\widehat{\sigma}^2_{2,N},\alpha)=0,$$

i.e.

$$\sum_{(x_1,x_2)\in E_N}\frac{\partial}{\partial\sigma}\log f_{x_1,x_2}^{(\widehat{\sigma}_{2,N})}(\eta(x_1),\eta(x_2))=0.$$

Proposition

Let $u\in\mathbf{R}$, $x_1,x_2\in\mathbf{R}^2$ and $z_1,z_2\in\mathbf{R}_+$ such that $d^{-\alpha/2}\sigma^{-1}\log(z_2/z_1)=u$, with $d=\|x_2-x_1\|>0$. Then

$$\lim_{d\to 0}\frac{\partial}{\partial\sigma}\log f_{x_1,x_2}^{(\sigma)}(z_1,z_2)=\frac{1}{\sigma}(u^2-1).$$

Reformulation of the main theorem

Notation:

- normalized increment (roughly standard Gaussian):

$$U^{(\eta)}_{x_1,x_2} = \frac{1}{\sigma d_{12}^{\alpha/2}} \log\left(\frac{\eta(x_2)}{\eta(x_1)}\right);$$

- square increment sums:

$$V^{(\eta)}_{2,N} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \Big((U^{(\eta)}_{x_1,x_2})^2 - 1 \Big).$$

Theorem

Let $\alpha \in (0,1)$. Then

$$N^{-(2-\alpha)/4} V^{(\eta)}_{2,N} \xrightarrow{\mathbb{P}} c \sum_{j\geq 1} \sum_{k>j} L_{Z_{k\setminus j}}(0).$$

Step 1. CLT for one trajectory

Notation:

- normalized increment of W:

$$U^{(W)}_{x_1,x_2} = \frac{1}{\sigma d_{12}^{\alpha/2}}(W(x_2) - W(x_1));$$

- square increment sums:

$$V^{(W)}_{2,N} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \left((U^{(W)}_{x_1,x_2})^2 - 1 \right).$$

Proposition

Let $\alpha \in (0,1)$. Then

$$V^{(W)}_{2,N} \xrightarrow{\mathscr{D}} \mathcal{N}(0, \sigma^2_{V_2}).$$

Step 2. Local time for two trajectories

Notation:

- normalized increment of $W^{(1)} \vee W^{(2)}$ (independent random fields):

$$U_{x_1,x_2}^{(W_\vee)} = \frac{1}{\sigma d_{12}^{\alpha/2}} \Big(W^{(1)} \vee W^{(2)}(x_2) - W^{(1)} \vee W^{(2)}(x_1) \Big);$$

- square increment sums:

$$V_{2,N}^{(W_\vee)} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \Big((U_{x_1,x_2}^{(W_\vee)})^2 - 1 \Big).$$

Proposition

Let $\alpha \in (0,1)$. Then

$$N^{-(2-\alpha)/4} V_{2,N}^{(W_\vee)} \xrightarrow{\mathbb{P}} c\, L_{W^{(2)}-W^{(1)}}(0).$$

Proof for two trajectories (1)

Notation:

- $W^{(2\backslash 1)}(x) = W^{(2)}(x) - W^{(1)}(x)$;
- $U^{(i)}_{x_1,x_2}$: increment for $W^{(i)}$.

Decomposition:

$$V^{(W_\vee)}_{2,N} = V^{(1)}_{2,N} + V^{(2)}_{2,N} + V^{(2/1)}_{2,N},$$

where

$$V^{(1)}_{2,N} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N, W^{(2\backslash 1)}(x_1)<0} \left((U^{(1)}_{x_1,x_2})^2 - 1 \right)$$

$$V^{(2)}_{2,N} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N, W^{(2\backslash 1)}(x_1)>0} \left((U^{(2)}_{x_1,x_2})^2 - 1 \right)$$

$$V^{(2/1)}_{2,N} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \Psi\left(U^{(1)}_{x_1,x_2}, U^{(2)}_{x_1,x_2}, W^{(2\backslash 1)}(x_1)/d_{12}^{\alpha/2} \right).$$

Proof for two trajectories (2)

Dominant term:

$$\begin{aligned} V_{2,N}^{(2/1)} &= \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \Psi\left(U^{(1)}_{x_1,x_2}, U^{(2)}_{x_1,x_2}, W^{(2\backslash 1)}(x_1)/d_{12}^{\alpha/2}\right) \\ &\simeq \frac{c}{\sqrt{|E_N|}} \sum_{x\in P_N\cap[0,1]^2} F\left(N^{\alpha/4} W^{(2\backslash 1)}(x)\right) \\ &\simeq \frac{cN^{-\alpha/4}}{\sqrt{|E_N|}} \sum_{x\in P_N\cap[0,1]^2} \int_{\mathbf{R}}\int_{\mathbf{R}} F(y) e^{\mathrm{i}\xi\left(W^{(2\backslash 1)}(x)-N^{-\alpha/4}y\right)} \mathrm{d}y\mathrm{d}\xi \\ &\simeq cN^{(2-\alpha)/4} L_{W^{(2\backslash 1)}}(0) \end{aligned}$$

Consequence:

$$V_{2,N}^{(W_\vee)} \simeq c_1 + c_2 + cN^{(2-\alpha)/4} L_{W^{(2\backslash 1)}}(0) \simeq cN^{(2-\alpha)/4} L_{W^{(2\backslash 1)}}(0).$$

Step 3. Local times for Brown-Resnick

Notation:

- normalized increment (roughly standard Gaussian):

$$U^{(\eta)}_{x_1,x_2} = \frac{1}{\sigma d_{12}^{\alpha/2}} \log\left(\frac{\eta(x_2)}{\eta(x_1)}\right);$$

- square increment sums:

$$V^{(\eta)}_{2,N} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \Big((U^{(\eta)}_{x_1,x_2})^2 - 1 \Big).$$

Theorem

Let $\alpha \in (0,1)$. Then

$$N^{-(2-\alpha)/4} V^{(\eta)}_{2,N} \xrightarrow{\mathbb{P}} c \sum_{j\geq 1} \sum_{k>j} L_{Z_{k\setminus j}}(0).$$

Proof

- Given $(x_1, x_2) \in E_N$, it is likely that $x_1, x_2 \in \mathbf{C}_{k,j}$ for some j, k.

- Therefore

$$V_{2,N}^{(\eta)} \simeq \sum_{j\geq 1} \sum_{k>j} \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \left((U_{x_1,x_2}^{(\eta)})^2 - 1 \right) \mathbb{1}_{x_1,x_2 \in \mathbf{C}_{k,j}}$$

$$\simeq \sum_{j\geq 1} \sum_{k>j} \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \Psi\left(U_{x_1,x_2}^{(W_j)}, U_{x_1,x_2}^{(W_k)}, Z_{k\setminus j}(x_1)/d_{12}^{\alpha/2} \right) \mathbb{1}_{x_1,x_2 \in \mathbf{C}_{k,j}}$$

$$\simeq cN^{-(2-\alpha)/4} \sum_{j\geq 1} \sum_{k>j} L_{Z_{k\setminus j}}(0).$$

Perspectives

- $\alpha \geq 1$?
- (σ^2, α) simultaneously?
- higher dimension?
- other max-stable random fields?