Texture Modeling:
Self-Similar Gaussian Fields and Monogenic Signal

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Stochastic Geometry Days
Goal: **Texture modeling and analysis**

- Exemplar-based texture synthesis

Exemplar texture  \( H = 0.1 \)  \( H = 0.5 \)  \( H = 0.9 \)

- Medical diagnostic: e.g. difference between dense and fatty breast tissue\(^1\)

Self-similar random fields using the **monogenic signal**: Riesz transform.

\(^1\)BIRADS database
1. Self-similar Gaussian random fields

2. Monogenic signal and multiscale analysis

3. Monogenic parameters for anisotropy and self-similarity estimation
Harmonizable random fields with stationary increments

Given $W$ an isotropic complex Gaussian measure and $f$ a spectral density function, the random field defined for all $x \in \mathbb{R}^2$, by

$$X(x) := \mathbb{R} \left( \int [e^{-i x \cdot \xi} - 1] \sqrt{f(\xi)} W(d\xi) \right)$$

is a Gaussian random field with stationary increments, meaning that

$$\forall \ x_0 \in \mathbb{R}^2, \{X(x + x_0) - X(x); x \in \mathbb{R}^2\} \overset{d}{=} \{X(x); x \in \mathbb{R}^2\}.$$
Harmonizable random fields with stationary increments

Given $W$ an isotropic complex Gaussian measure and $f$ a spectral density function, the random field defined for all $x \in \mathbb{R}^2$, by

$$X(x) := \Re \left( \int [e^{-ix \cdot \xi} - 1] \sqrt{f(\xi)} W(d\xi) \right)$$

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Given a function $u : \mathbb{R}^2 \to \mathbb{R}$ such that $|\hat{u}(\xi)| \leq C \min(1, |\xi|)$, define the generalized field

$$\langle X, u \rangle := \Re \left( \int_{\mathbb{R}^2} \hat{u}(\xi) \sqrt{f(\xi)} W(d\xi) \right).$$
Harmonizable random fields with stationary increments

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Given a function $u : \mathbb{R}^2 \to \mathbb{R}$ such that $|\hat{u}(\xi)| \leq C \min(1, |\xi|)$, define the generalized field

$$\langle X, u \rangle := \Re \left( \int_{\mathbb{R}^2} \hat{u}(\xi) \sqrt{f(\xi)} W(d\xi) \right).$$

Then, $\text{Cov}(\langle X, u \rangle, \langle X, v \rangle) = \Re \left( \int_{\mathbb{R}^2} \hat{u}(\xi) \hat{v}(\xi) f(\xi) d\xi \right)$ and for $X$ a Gaussian random field with stationary increment,

$$\langle X, u \rangle \sim \mathcal{N} \left( 0, \int_{\mathbb{R}^2} |\hat{u}(\xi)|^2 f(\xi) d\xi \right).$$
Self-similar random field
For $H \in (0, 1)$ and a spectral density $f : \xi \mapsto t \left( \frac{\xi}{|\xi|} \right) |\xi|^{-2H-2}$, with $t \in L^1(S^1)$ is defined on the unit sphere, even and positive, such that $f \in L^1(\mathbb{R}^2, \min(1, |\xi|^2 \, d\xi))$. Then, the random field $X$ associated to $f$ is self-similar of order $H$, meaning that

$$\{X(\lambda x); x \in \mathbb{R}^2 \} \overset{d}{=} \lambda^H \{X(x); x \in \mathbb{R}^2 \}.$$ 

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Self-similar random fields with stationary increments (SSI)

Self-similar random field
For $H \in (0, 1)$ and a spectral density $f : \xi \mapsto t \left( \frac{\xi}{|\xi|} \right) |\xi|^{-2H-2}$, with $t \in L^1(S^1)$ is defined on the unit sphere, even and positive, such that $f \in L^1(\mathbb{R}^2, \min(1, |\xi|^2 \, d\xi))$. Then, the random field $X$ associated to $f$ is self-similar of order $H$, meaning that

$$\{X(\lambda x); x \in \mathbb{R}^2\} \overset{d}{=} \lambda^H \{X(x); x \in \mathbb{R}^2\}.$$ 

Example: Elementary fields \(^2\) Consider $\delta \in (0, \frac{\pi}{2}]$ and $t_\delta(\alpha) = 1_{|\alpha| \leq \delta}$ for $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. For $\delta = \frac{\pi}{2}$, $X$ is the isotropic fractional Brownian field.

Riesz transforms\(^3\) Given a signal \(s \in L^2(\mathbb{R}^2)\), the Riesz transforms are defined, for \(k = 1, 2\) by

\[
\mathcal{R}_k(s)(x) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon(x)} \frac{x_k - y_k}{|x - y|^3} s(y) dy.
\]

(1)

Then,

\[
\hat{\mathcal{R}_k}(s)(\xi) = -i \frac{\xi_k}{|\xi|} \hat{s}(\xi) \text{ for } \xi \in \mathbb{R}^2.
\]

The associated \textbf{monogenic signal} \(s_M(x)\) is a signal in \(\mathbb{R}^3\)

\[
s_M(x) = \begin{bmatrix}
s(x) \\ \mathcal{R}_1(s)(x) \\ \mathcal{R}_2(s)(x)
\end{bmatrix}
\]

\(^3\)M. Felsberg et G. Sommer, \textit{The monogenic signal}, 2001
Riesz transforms Given a signal $s \in L^2(\mathbb{R}^2)$, the Riesz transforms are defined, for $k = 1, 2$ by

$$R_k(s)(x) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon(x)} \frac{x_k - y_k}{|x - y|^3} s(y) dy.$$  \hspace{1cm} (1)

Then,

$$\widehat{R_k(s)}(\xi) = -i \frac{\xi_k}{|\xi|} \widehat{s}(\xi) \text{ for } \xi \in \mathbb{R}^2.$$

The associated monogenic signal $s_M(x)$ is a signal in $\mathbb{R}^3$ and can also be characterized by its spherical coordinates:

$$s_M(x) = \begin{bmatrix} s(x) \\ R_1(s)(x) \\ R_2(s)(x) \end{bmatrix} = \begin{bmatrix} A(x) \cos \varphi(x) \\ A(x) \sin \varphi(x) \cos \theta(x) \\ A(x) \sin \varphi(x) \sin \theta(x) \end{bmatrix},$$

with $A(x) \in \mathbb{R}^+$ the amplitude of the monogenic signal, $\varphi(x) \in [0, \pi)$ its phase and $\theta(x) \in [-\pi, \pi)$ its orientation.

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\(^3\)M. Felsberg et G. Sommer, *The monogenic signal*, 2001
Riesz transforms and monogenic signal

Advantages of the Riesz transform:

- Similar to the Gradient operator: detection of contours and orientations
- Very easy to compute in the Fourier domain
- Adapted to a multiscale analysis\(^4\)

Monogenic representation of a random field

**Monogenic signal:** Given a Gaussian random field $X$ and a function $u \in S_0(\mathbb{R}^2)$, the monogenic signal is

$$MX(u) = \langle X, u \rangle_M = \begin{bmatrix} \langle X, u \rangle \\ \langle R_1 X, u \rangle \\ \langle R_2 X, u \rangle \end{bmatrix} = (\langle X, u \rangle, R_X(u)).$$

It is a Gaussian vector and its covariance function is

$$C_{MX}(u, v) := \mathbb{E}(MX(u)MX(v)^*) = \Re \int \begin{pmatrix} 1 & i \frac{\xi_1}{|\xi|} & i \frac{\xi_2}{|\xi|} \\ -i \frac{\xi_1}{|\xi|} & \frac{\xi_1^2}{|\xi|^2} & \frac{\xi_1 \xi_2}{|\xi|^2} \\ -i \frac{\xi_2}{|\xi|} & \frac{\xi_1 \xi_2}{|\xi|^2} & \frac{\xi_2^2}{|\xi|^2} \end{pmatrix} \hat{u}(\xi)\overline{\hat{v}(\xi)}f(\xi) d\xi.$$
We consider a multiscale representation of the initial random field using a function \( u \), such that \( u \in S_0(\mathbb{R}^2) \), with \( \int_{\mathbb{R}^2} u(x) dx = 1 \) and \( u_j(x) = 2^{-j} u(2^{-j} x) \).

Note that in our experiments, we used an \textbf{undecimated filter bank}, given by

\[
\begin{align*}
\hat{G}_1(\xi) &= 1 - e^{-|\xi|^2/2}, \\
\hat{G}_j(\xi) &= \hat{G}_1(2^{j-1} \xi), \\
\hat{H}_j(\xi) &= \sqrt{1 - \hat{G}_j(\xi)^2}.
\end{align*}
\]  

At each scale \( j \), we consider the Riesz transform of the filtered random field

\[
\mathcal{R}_X(u_j) = (\langle \mathcal{R}_1 X, u_j \rangle, \langle \mathcal{R}_2 X, u_j \rangle),
\]

and we denote \( \tau_x u_j \) the translation of \( u_j \) in \( x \), used to evaluate the random field in \( x \).
Proposition
Consider \((\delta, H) \in (0, \pi/2] \times (0, 1)\) and \(u \in S_0(\mathbb{R}^2)\) a radial function. If \(X\) is an elementary field with spectral density \(f_X(\xi) = t_\delta(\xi/|\xi|)|\xi|^{-2H-2}\) then \((MX(\tau_x u))_{x \in \mathbb{Z}^2}\) is a centered stationary Gaussian field such that for all \(x \in \mathbb{Z}^2\),

\[
MX(\tau_x u) \overset{d}{=} \sqrt{c_X(u)} D_\delta Z,
\]

where

- \(c_X(u) = \text{Var}(\langle X, u \rangle) = \int_{\mathbb{R}^2} |\hat{u}(\xi)|^2 f(\xi) d\xi\),
- \(D_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2} + \frac{\sin(2\delta)}{4\delta}} & 0 \\ 0 & 0 & \sqrt{\frac{1}{2} - \frac{\sin(2\delta)}{4\delta}} \end{pmatrix}\),
- \(Z \sim \mathcal{N}(0, I_3)\).

Moreover, for a given scale \(j\),

\[
(My(\tau_x u_j))_{x \in \mathbb{Z}^2} \overset{d}{=} 2^{j(H+1)} (MX(\tau_x u))_{x \in \mathbb{Z}^2}.
\]
Riesz structure tensor\textsuperscript{5}:

\[
J_X(u_j) := \mathbb{E} (R_X(u_j)R_X(u_j)^*) ,
\]

with \((\lambda^+(u_j), \lambda^-(u_j))\) its largest and smallest eigenvalues.

Coherence index:

\[
\chi_X(u_j) = \frac{\lambda^+(u_j) - \lambda^-(u_j)}{\lambda^+(u_j) + \lambda^-(u_j)} \in [0, 1),
\]

allows to measure the \textbf{directional anisotropy}.

\textsuperscript{5}K. Polisano. \textit{Modélisation de textures anisotropes par la transformée en ondelettes monogéniques}, 2017
Riesz structure tensor\(^5\):

\[ J_X(u_j) := \mathbb{E}(RX(u_j)RX(u_j)^*) , \]

with \((\lambda^+(u_j), \lambda^-(u_j))\) its largest and smallest eigenvalues.

Coherence index:

\[ \chi_X(u_j) = \frac{\lambda^+(u_j) - \lambda^-(u_j)}{\lambda^+(u_j) + \lambda^-(u_j)} \in [0, 1) , \]

allows to measure the \textbf{directional anisotropy}.

For an elementary field,

\[ \lambda^\pm(u_j) = 2^j(2^H+2)\lambda^\pm(u) \text{ and } \chi_X(u_j) = \frac{\sin(2\delta)}{2\delta} . \quad (3) \]

\(^5\)K. Polisano. \textit{Modélisation de textures anisotropes par la transformée en ondelettes monogéniques}, 2017
Distribution of the spherical coordinates of $MX(u)$

**Proposition** • $(A(\tau_x u), \theta(\tau_x u), \varphi(\tau_x u))_{x \in \mathbb{Z}^2}$ is a stationary field.

• **Orientation distribution:** $\theta(u)$ is independent of $(A(u), \varphi(u))$ and follows an offset normal distribution whose probability density function is $\pi$-periodic and given by

$$t \mapsto \frac{\sqrt{1 - \chi_X(u)^2}}{2\pi(1 - \chi_X(u)\cos(2t))},$$

where $\chi_X(u) \in [0, 1)$ is the coherence index.

• **Isotropic case:** In the isotropic case, if $\delta = \pi/2$, $\theta(u)$ follows a uniform distribution on $(-\pi, \pi)$ and the density function of the phase $\varphi(u)$ is given by

$$\phi \mapsto \frac{|\sin(\phi)|}{(1 + \sin(\phi)^2)^{3/2}} 1_{(0, \pi)}(\phi).$$
Monogenic representation of an elementary field

Elementary field with \( H = 0.5, \delta = \pi/2 \).
Monogenic representation of an elementary field

Random field

Filtered field

Riesz 1

Riesz 2

Amplitude

Elementary field with $H = 0.5$, $\delta = \pi/3$. 

Phase

Orientation
Monogenic representation of an elementary field

Random field

Filtered field

Riesz 1

Riesz 2

Amplitude

Phase

Orientation

Elementary field with $H = 0.5, \delta = \pi/6.$
**Empirical estimator** of the structure tensor:

\[
J_{j}^{\text{emp}} = \frac{1}{N^2} \sum_{x \in G} R_X(\tau_x u_j) R_X(\tau_x u_j)^*,
\]

with \((\lambda^+_{\text{emp}}(u_j), \lambda^-_{\text{emp}}(u_j))\) its largest and smallest eigenvalues.

**Proposition** \(J_{j}^{\text{emp}}\) is an unbiased estimator and

\[
J_{j}^{\text{emp}} \xrightarrow{a.s.} J_X(u_j). \tag{4}
\]

Besides, \(\lambda^\pm_{j}^{\text{emp}} \xrightarrow{a.s.} \lambda^\pm_{j} \).

\[N \to \infty\]
Inference using the structure tensor - Coherence index

\[ \chi_x(u_j) = \frac{\sin(2\delta)}{2\delta} \]

Coherence index estimation, for 1000 realizations, depending on a) the scale of the monogenic representation \( j \) and the degree of anisotropy \( \delta \), with \( H = 0.5 \) and b) the degree of anisotropy and \( H \), with \( j = 3 \).
Inference using the structure tensor - Hurst index

\[ \lambda^\pm(u_j) = 2^{j(2H+2)} \lambda^\pm(u) \]

**a)**  \( \lambda_j^\pm \text{emp} \)

**b)** Estimation of \( H \)

Estimation of \( H \), obtained by the eigenvalues \( \lambda_j^\pm \text{emp} \) from \( J_X(u_j) \) (a). Dotted lines are computed by linear regression and provide an estimator of the Hurst parameter \( H \) (b).
Inference using the monogenic signal

**Proposition**

**Squared amplitude:** \( A(u_j)^2 = \langle X, u_j \rangle^2 + |RX(u_j)|^2 \) and \( A(u_j)^2 \overset{d}{=} c_X(u_j)A^2 \), with \( A^2 = |D_\delta Z|^2 \) for \( Z \sim \mathcal{N}(0, I_3) \), \( D_\delta \) and

\[
\begin{align*}
c_X(u_j) &= \text{Var}(\langle X, u \rangle) = 2^{j(2H+2)} \text{Var}(\langle X, u \rangle).
\end{align*}
\]

The estimator \( V_j^{\text{emp}} = \frac{1}{N^2} \sum_{x \in G} |MX(\tau_x u_j)|^2 \) is unbiased and

\[
V_j^{\text{emp}} \xrightarrow{\text{a.s.}} \frac{1}{N \to \infty} \mathbb{E} \left( |MX(u_j)|^2 \right).
\]
Inference using the monogenic signal

**Proposition**

**Squared amplitude:** \( A(u_j)^2 = \langle X, u_j \rangle^2 + |RX(u_j)|^2 \) and \( A(u_j) \stackrel{d}{=} c_X(u_j)A^2 \), with \( A^2 = |D_\delta Z|^2 \) for \( Z \sim \mathcal{N}(0, I_3) \), \( D_\delta \) and

\[
 c_X(u_j) = \text{Var}(\langle X, u_j \rangle) = 2^j(2^{H+2})\text{Var}(\langle X, u \rangle).
\]

The estimator \( V_{j}^{\text{emp}} = \frac{1}{N^2} \sum_{x \in G} |MX(\tau_x u_j)|^2 \) is unbiased and

\[
 V_{j}^{\text{emp}} \xrightarrow{a.s.} N \rightarrow \infty \mathbb{E}(|MX(u_j)|^2).
\]

c) \( V_{j}^{\text{emp}} \)

d) Estimation of \( H \)
Conclusion

- Two methods based on the Riesz transform to estimate the parameters of elementary fields.
- Using the full monogenic signal seems to produce more stable estimators.

Perspectives:

- Study further the statistical properties of these estimators
- Get similar results for larger categories of self-similar random fields
- Extend to color textures ie multivariate random fields

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\(^6\) C.L., H. Biermé, C. Lacaux, P. Carré, Modélisation de Textures : Champs Gaussiens Autosimilaires et Signal Monogène, 2023 (accepted)