Texture Modeling:
Self-Similar Gaussian Fields and Monogenic Signal

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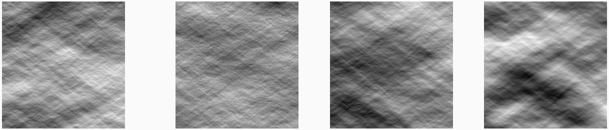
Introduction

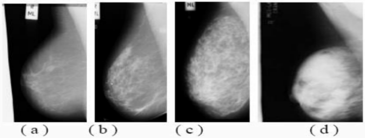
Goal: **Texture modeling and analysis**

- Exemplar-based texture synthesis



Exemplar texture $H = 0.1$ $H = 0.5$ $H = 0.9$

- Medical diagnostic: e.g. difference between dense and fatty breast tissue[1]



Self-similar random fields using the **monogenic signal**: Riesz transform.

[1] BIRADS database

Outline

1. Self-similar Gaussian random fields
2. Monogenic signal and multiscale analysis
3. Monogenic parameters for anisotropy and self-similarity estimation

Harmonizable random fields with stationary increments

Given W an isotropic complex Gaussian measure and f a spectral density function, the random field defined for all $x \in \mathbb{R}^2$, by

$$X(x) := \Re\left(\int [e^{-ix\cdot\xi} - 1]\sqrt{f(\xi)}W(d\xi)\right)$$

is a Gaussian random field with stationary increments, meaning that

$$\forall x_0 \in \mathbb{R}^2, \{X(x+x_0) - X(x); x \in \mathbb{R}^2\} \overset{d}{=} \{X(x); x \in \mathbb{R}^2\}.$$

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Given a function $u : \mathbb{R}^2 \to \mathbb{R}$ such that $|\widehat{u}(\xi)| \leq C\min(1, |\xi|)$, define the generalized field

$$\langle X, u\rangle := \Re\left(\int_{\mathbb{R}^2} \widehat{u}(\xi)\sqrt{f(\xi)}W(d\xi)\right).$$

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$$\langle X, u\rangle := \Re\left(\int_{\mathbb{R}^2} \widehat{u}(\xi)\sqrt{f(\xi)}W(d\xi)\right).$$

Then, $\mathrm{Cov}(\langle X, u\rangle, \langle X, v\rangle) = \Re\left(\int_{\mathbb{R}^2} \widehat{u}(\xi)\widehat{v}(\xi)f(\xi)d\xi\right)$ and for X a Gaussian random field with stationary increment,

$$\langle X, u\rangle \sim \mathcal{N}\left(0, \int_{\mathbb{R}^2} |\widehat{u}(\xi)|^2 f(\xi)d\xi\right).$$

Self-similar random fields with stationary increments (SSI)

Self-similar random field

For $H \in (0,1)$ and a spectral density $f : \xi \mapsto t\left(\frac{\xi}{|\xi|}\right)|\xi|^{-2H-2}$, with $t \in L^1(S^1)$ is defined on the unit sphere, even and positive, such that $f \in L^1(\mathbb{R}^2, \min(1, |\xi|^2 d\xi))$. Then, the random field X associated to f is self-similar of order H, meaning that

$$\{X(\lambda x); x \in \mathbb{R}^2\} \overset{d}{=} \lambda^H \{X(x); x \in \mathbb{R}^2\}.$$

[2] Biermé, Moisan, Richard, *A Turning-Band Method for the Simulation of Anisotropic Fractional Brownian Fields*, 2015

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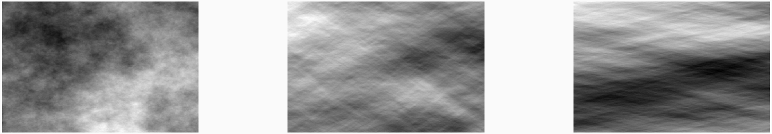
Example: Elementary fields[2] Consider $\delta \in (0, \frac{\pi}{2}]$ and $t_\delta(\alpha) = 1_{|\alpha| \leq \delta}$ for $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. For $\delta = \frac{\pi}{2}$, X is the isotropic fractional Brownian field.



$\delta = \frac{\pi}{2}$ $\delta = \frac{\pi}{3}$ $\delta = \frac{\pi}{6}$

[2] Biermé, Moisan, Richard, *A Turning-Band Method for the Simulation of Anisotropic Fractional Brownian Fields*, 2015

Riesz transforms and monogenic signal

Riesz transforms[3] Given a signal $s \in L^2(\mathbb{R}^2)$, the Riesz transforms are defined, for $k = 1, 2$ by

$$\mathcal{R}_k(s)(x) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon(x)} \frac{x_k - y_k}{|x - y|^3} s(y) dy. \quad (1)$$

Then,

$$\widehat{\mathcal{R}_k(s)}(\xi) = -i \frac{\xi_k}{|\xi|} \widehat{s}(\xi) \text{ for } \xi \in \mathbb{R}^2.$$

The associated **monogenic signal** $s_M(x)$ is a signal in $\mathbb{R}^3$

$$s_M(x) = \begin{bmatrix} s(x) \\ \mathcal{R}_1(s)(x) \\ \mathcal{R}_2(s)(x) \end{bmatrix}$$

[3] M. Felsberg et G. Sommer, *The monogenic signal*, 2001

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The associated **monogenic signal** $s_M(x)$ is a signal in $\mathbb{R}^3$ and can also be characterized by its spherical coordinates:

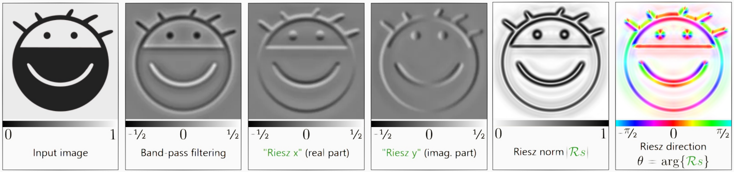
$$s_M(x) = \begin{bmatrix} s(x) \\ \mathcal{R}_1(s)(x) \\ \mathcal{R}_2(s)(x) \end{bmatrix} = \begin{bmatrix} A(x) \cos \varphi(x) \\ A(x) \sin \varphi(x) \cos \theta(x) \\ A(x) \sin \varphi(x) \sin \theta(x) \end{bmatrix},$$

with $A(x) \in \mathbb{R}^+$ the amplitude of the monogenic signal, $\varphi(x) \in [0, \pi)$ its phase and $\theta(x) \in [-\pi, \pi)$ its orientation.

[3] M. Felsberg et G. Sommer, *The monogenic signal*, 2001

Riesz transforms and monogenic signal

Advantages of the Riesz transform:



- Similar to the Gradient operator: detection of contours and orientations
- Very easy to compute in the Fourier domain
- Adapted to a multiscale analysis[4]

[4] R. Soulard et P. Carré, *Characterization of color images with multiscale monogenic maxima*, 2018. Images from R. Soulard's website.

Monogenic representation of a random field

Monogenic signal: Given a Gaussian random field X and a function $u \in \mathcal{S}_0(\mathbb{R}^2)$, the monogenic signal is

$$MX(u) = \langle X, u \rangle_M = \begin{bmatrix} \langle X, u \rangle \\ \langle \mathcal{R}_1 X, u \rangle \\ \langle \mathcal{R}_2 X, u \rangle \end{bmatrix} = \left(\langle X, u \rangle, \mathcal{R}_X(u) \right).$$

It is a Gaussian vector and its covariance function is

$$\mathcal{C}_{MX}(u, v) := \mathbb{E}\left(MX(u) MX(v)^* \right) = \Re \int \begin{pmatrix} 1 & i\frac{\xi_1}{|\xi|} & i\frac{\xi_2}{|\xi|} \\ -i\frac{\xi_1}{|\xi|} & \frac{\xi_1^2}{|\xi|^2} & \frac{\xi_1\xi_2}{|\xi|^2} \\ -i\frac{\xi_2}{|\xi|} & \frac{\xi_1\xi_2}{|\xi|^2} & \frac{\xi_2^2}{|\xi|^2} \end{pmatrix} \hat{u}(\xi)\overline{\hat{v}(\xi)} f(\xi) d\xi.$$

Multiscale representation

We consider a multiscale representation of the initial random field using a function u, such that $u \in S_0(\mathbb{R}^2)$, with $\int_{\mathbb{R}^2} u(x)dx = 1$ and $u_j(x) = 2^{-j}u(2^{-j}x)$.

Note that in our experiments, we used an **undecimated filter bank**, given by

$$\begin{cases} \widehat{G}_1(\xi) = 1 - e^{-|\xi|^2/2}, \\ \widehat{G}_j(\xi) = \widehat{G}_1(2^{j-1}\xi), \\ \widehat{H}_j(\xi) = \sqrt{1 - \widehat{G}_j(\xi)^2}. \end{cases} \tag{2}$$

At each scale j, we consider the Riesz transform of the filtered random field

$$\mathcal{R}_X(u_j) = \left(\langle \mathcal{R}_1 X, u_j \rangle, \langle \mathcal{R}_2 X, u_j \rangle\right),$$

and we denote $\tau_x u_j$ the translation of u_j in x, used to evaluate the random field in x.

Monogenic representation of an elementary field

Proposition
Consider $(\delta, H) \in (0, \pi/2] \times (0, 1)$ and $u \in \mathcal{S}_0(\mathbb{R}^2)$ a radial function. If X is an elementary field with spectral density $f_X(\xi) = t_\delta(\xi/|\xi|)|\xi|^{-2H-2}$ then $(MX(\tau_x u))_{x \in \mathbb{Z}^2}$ is a centered stationary Gaussian field such that for all $x \in \mathbb{Z}^2$,

$$MX(\tau_x u) \overset{d}{=} \sqrt{c_X(u)} D_\delta Z,$$

where

- $c_X(u) = \mathrm{Var}(\langle X, u \rangle) = \int_{\mathbb{R}^2} |\widehat{u}(\xi)|^2 f(\xi) d\xi$,
- $D_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2} + \frac{\sin(2\delta)}{4\delta}} & 0 \\ 0 & 0 & \sqrt{\frac{1}{2} - \frac{\sin(2\delta)}{4\delta}} \end{pmatrix}$
- $Z \sim \mathcal{N}(0, I_3)$.

Moreover, for a given scale j,

$$(MX(\tau_x u_j))_{x \in \mathbb{Z}^2} \overset{d}{=} 2^{j(H+1)} (MX(\tau_x u))_{x \in \mathbb{Z}^2}.$$

Riesz structure tensor and coherence index

Riesz structure tensor[5]**:**

$$J_X(u_j) := \mathbb{E}\left(\mathcal{R}X(u_j)\mathcal{R}X(u_j)^*\right),$$

with $(\lambda^+(u_j), \lambda^-(u_j))$ its largest and smallest eigenvalues.

Coherence index:

$$\chi_X(u_j) = \frac{\lambda^+(u_j) - \lambda^-(u_j)}{\lambda^+(u_j) + \lambda^-(u_j)} \in [0, 1),$$

allows to measure the **directional anisotropy**.

[5] K. Polisano. *Modélisation de textures anisotropes par la transformée en ondelettes monogéniques*, 2017

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allows to measure the **directional anisotropy**.

For an elementary field,

$$\lambda^{\pm}(u_j) = 2^{j(2H+2)}\lambda^{\pm}(u) \text{ and } \chi_X(u_j) = \frac{\sin(2\delta)}{2\delta}. \qquad (3)$$

[5] K. Polisano. *Modélisation de textures anisotropes par la transformée en ondelettes monogéniques*, 2017

Distribution of the spherical coordinates of $MX(u)$

Proposition • $(A(\tau_x u), \theta(\tau_x u), \varphi(\tau_x u))_{x \in \mathbb{Z}^2}$ is a stationary field.

• **Orientation distribution:** $\theta(u)$ is independent of $(A(u), \varphi(u))$ and follows an offset normal distribution whose probability density function is π-periodic and given by

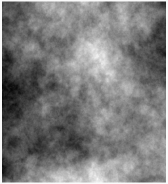
$$t \mapsto \frac{\sqrt{1 - \chi_X(u)^2}}{2\pi(1 - \chi_X(u)\cos(2t))},$$

where $\chi_X(u) \in [0, 1)$ is the coherence index.

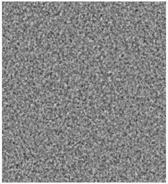
• **Isotropic case:** In the isotropic case, if $\delta = \pi/2$, $\theta(u)$ follows a uniform distribution on $(-\pi, \pi)$ and the density function of the phase $\varphi(u)$ is given by

$$\phi \mapsto \frac{|\sin(\phi)|}{(1 + \sin(\phi)^2)^{3/2}} 1_{(0,\pi)}(\phi).$$

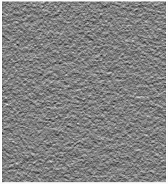
Monogenic representation of an elementary field



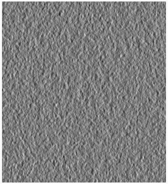
Random field



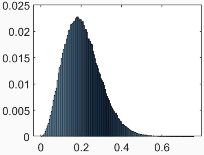
Filtered field



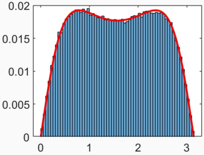
Riesz 1



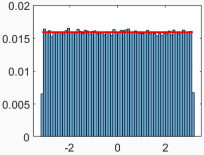
Riesz 2



Amplitude

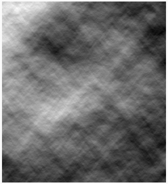


Phase

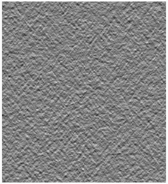


Orientation

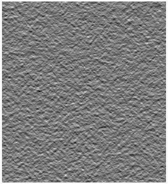
Elementary field with $H = 0.5, \delta = \pi/2$.



Random field



Filtered field



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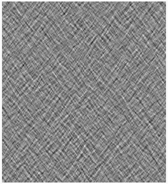
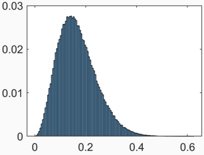
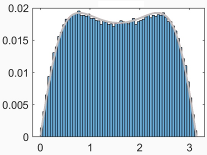


Riesz 2



Amplitude



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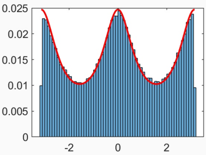
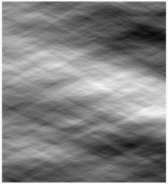


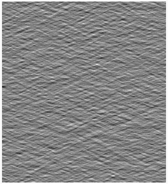
Orientation

Elementary field with $H = 0.5, \delta = \pi/3$.

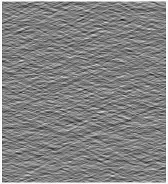
Monogenic representation of an elementary field



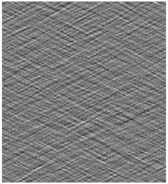
Random field



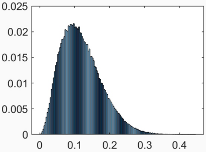
Filtered field



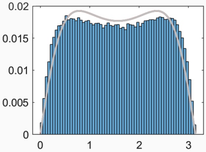
Riesz 1



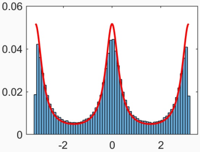
Riesz 2



Amplitude



Phase



Orientation

Elementary field with $H = 0.5, \delta = \pi/6$.

Inference using the structure tensor

Empirical estimator of the structure tensor:

$$J_j^{\mathbf{emp}} = \frac{1}{N^2} \sum_{x \in G} \mathcal{R}X(\tau_x u_j) \mathcal{R}X(\tau_x u_j)^*,$$

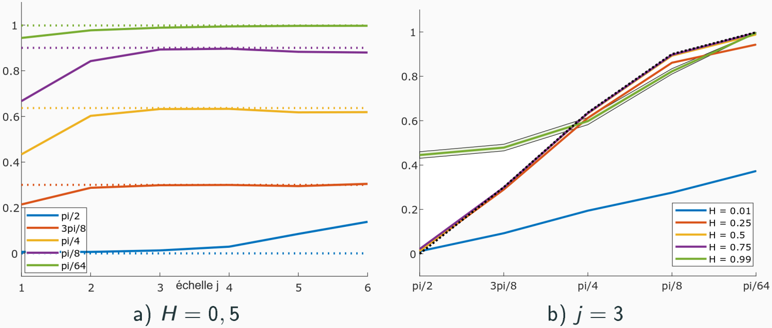
with $(\lambda^{+\mathbf{emp}}(u_j), \lambda^{-\mathbf{emp}}(u_j))$ its largest and smallest eigenvalues.

Proposition $J_j^{\mathbf{emp}}$ is an unbiased estimator and

$$J_j^{\mathbf{emp}} \xrightarrow[N \to \infty]{a.s.} J_X(u_j). \tag{4}$$

Besides, $\lambda_j^{\pm\mathbf{emp}} \xrightarrow[N \to \infty]{a.s.} \lambda_j^{\pm}$.

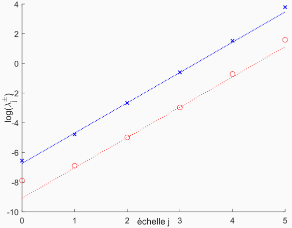
$$\chi_X(u_j) = \frac{\sin(2\delta)}{2\delta}$$



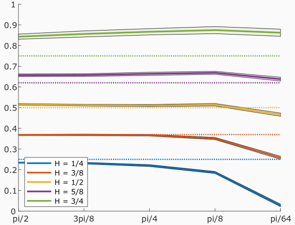
Coherence index estimation, for 1000 realizations, depending on a) the scale of the monogenic representation j and the degree of anisotropy δ, with $H = 0.5$ and b) the degree of anisotropy and H, with $j = 3$.

Inference using the structure tensor - Hurst index

$$\lambda^{\pm}(u_j) = 2^{j(2H+2)}\lambda^{\pm}(u)$$



a) $\lambda_j^{\pm \mathbf{emp}}$



b) Estimation of H

Estimation of H, obtained by the eigenvalues $\lambda_j^{\pm \mathbf{emp}}$ from $J_X(u_j)$ (a). Dotted lines are computed by linear regression and provide an estimator of the Hurst parameter H (b).

Inference using the monogenic signal

Proposition

Squared amplitude: $A(u_j)^2 = \langle X, u_j\rangle^2 + |\mathcal{R}X(u_j)|^2$ and $A(u_j)^2 \overset{d}{=} c_X(u_j)A^2$, with $A^2 = |D_\delta Z|^2$ for $Z \sim \mathcal{N}(0, I_3)$, D_δ and

$$c_X(u_j) = \mathrm{Var}(\langle X, u_j\rangle) = 2^{j(2H+2)}\mathrm{Var}(\langle X, u\rangle).$$

The estimator $V_j^{\mathrm{emp}} = \frac{1}{N^2}\sum_{x\in G}|MX(\tau_x u_j)|^2$ is unbiased and

$$V_j^{\mathrm{emp}} \xrightarrow[N\to\infty]{a.s.} \mathbb{E}\left(|MX(u_j)|^2\right).$$

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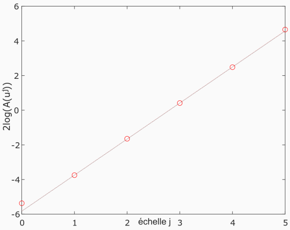
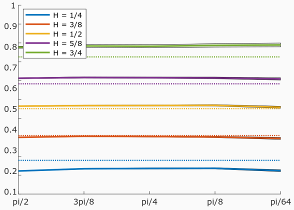


c) $V_j^{\mathbf{emp}}$



d) Estimation of H

Conclusion[6]

- Two methods based on the Riesz transform to estimate the parameters of elementary fields.
- Using the full monogenic signal seems to produce more stable estimators.

Perspectives:

- Study further the statistical properties of these estimators
- Get similar results for larger categories of self-similar random fields
- Extend to color textures ie multivariate random fields

[6] C.L., H. Biermé, C. Lacaux, P. Carré, *Modélisation de Textures : Champs Gaussiens Autosimilaires et Signal Monogène*, 2023 (accepted)