Unimodular Continuum Spaces

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Outline

1. The Mass Transport Principle in Various Subjects
2. Unimodular Continuum Spaces
3. Palm Theory
1. The Mass Transport Principle in Various Subjects

2. Unimodular Continuum Spaces

3. Palm Theory
Introduction

Connection between various fields:
- Stationary point processes,
- Unimodular random graphs,
- Unimodular discrete spaces,
- Stationary random measures,
- Scaling limits,
- Borel equivalence relations.

Key property: The **mass transport principle (MTP)**.
1. Point Processes

- \( \Phi \): A stationary point process on \( \mathbb{R}^d \).
  - i.e., a random discrete subset of \( \mathbb{R}^d \),
  - s.th., \( \forall t \in \mathbb{R}^d : \Phi + t \sim \Phi \).

- The **Palm version** of \( \Phi \):
  - \( \Phi_0 := \Phi \) conditioned on containing 0,
  - or \( \Phi \) seen from a *typical point of* \( \Phi \).
  - Formally:
    \[
    \mathbb{E} [h(\Phi_0)] = \frac{1}{\lambda} \mathbb{E} \left[ \sum_{x \in \Phi \cap [0,1]^d} h(\Phi - x) \right].
    \]
Mecke’s formula:
For all measurable functions \( h(\Phi_0, x) \geq 0 \) (for \( x \in \mathbb{R}^d \)):
\[
\mathbb{E} \left[ \sum_{x \in \Phi_0} h(\Phi_0, x) \right] = \mathbb{E} \left[ \sum_{x \in \Phi_0} h(\Phi_0 - x, -x) \right].
\]

Let \( g(\Phi_0, x, y) := h(\Phi_0 - x, y - x) \implies \)

---

**Theorem (MTP)**

*For all measurable functions \( g(\Phi_0, x, y) \geq 0 \) that are translation-invariant:*

\[
\mathbb{E} \left[ \sum_{x \in \Phi_0} g(\Phi_0, 0, x) \right] = \mathbb{E} \left[ \sum_{x \in \Phi_0} g(\Phi_0, x, 0) \right].
\]
2. Unimodular Graphs

- $\mathcal{G}_\star$: The space of all rooted graphs $(G, o)$ ($o \in V(G)$) up to isomorphisms.
- It is called unimodular if

$$
\mathbb{E} \left[ \sum_{x \in G} g(G, o, x) \right] = \mathbb{E} \left[ \sum_{x \in G} g(G, x, o) \right] \quad \text{(MTP)}
$$

for all measurable functions $g(G, x, y) \geq 0$ (for $x, y \in V(G)$) that are isometry-invariant.

- **Example:**
  1. Every finite graph $G$ with a uniformly-random root $o \in V(G)$.
  2. Cayley graphs.
  3. **Example:** Any graph constructed *equivariantly* on (the Palm version of) a stationary point process.
2. Unimodular Graphs

- \( G_* \): The space of all rooted graphs \((G, o) (o \in V(G))\) up to isomorphisms.

- \([G, o]\): A random rooted graph.

- It is called **unimodular** if

\[
\mathbb{E} \left[ \sum_{x \in G} g(G, o, x) \right] = \mathbb{E} \left[ \sum_{x \in G} g(G, x, o) \right] \quad \text{(MTP)}
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for all measurable functions \( g(G, x, y) \geq 0 \) (for \( x, y \in V(G) \)) that are isometry-invariant.

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  1. Every finite graph \( G \) with a uniformly-random root \( o \in V(G) \).
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- $G_\ast$: The space of all rooted graphs $(G, o) \ (o \in V(G))$ up to isomorphisms.
- It is called **unimodular** if

$$
\mathbb{E} \left[ \sum_{x \in G} g(G, o, x) \right] = \mathbb{E} \left[ \sum_{x \in G} g(G, x, o) \right] \quad \text{(MTP)}
$$

for all measurable functions $g(G, x, y) \geq 0$ (for $x, y \in V(G)$) that are isometry-invariant.

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1. Every finite graph $G$ with a uniformly-random root $o \in V(G)$.
2. Cayley graphs.
3. **Example**: Any graph constructed *equivariantly* on (the Palm version of) a stationary point process.
3. Unimodular Discrete Spaces

- \([D, o]\): A random rooted discrete metric space.
  - \(D\) should be *boundedly-finite*.
- It is called **unimodular** if for all measurable functions \(g(D, x, y) \geq 0\) (for \(x, y \in D\)) that are isometry-invariant,
  \[
  \mathbb{E} \left[ \sum_{x \in D} g(D, o, x) \right] = \mathbb{E} \left[ \sum_{x \in D} g(D, x, o) \right]. \tag{MTP}
  \]
- (Almost-) Unification of:
  - Unimodular graphs,
  - Palm version of stationary point processes,
  - Point-stationary point processes.
4. Random Measures

- $\Phi$: A stationary random measure on $\mathbb{R}^d$.
  - i.e., $\forall t \in \mathbb{R}^d : \Phi + t \sim \Phi$.

- **Example**: Every point process is a random measure.

- $\Phi_0$: The **Palm version** of $\Phi$,
  - or $\Phi$ seen from a *typical point*.
  - Heuristically:
    \[
    \mathbb{E}[g(\Phi_0, 0)] \leftrightarrow \int g(\Phi, x) d\Phi(x).
    \]

---

**Theorem (MTP)**

*For all measurable functions $g(\Phi_0, x, y) \geq 0$ that are translation-invariant:*

\[
\mathbb{E} \left[ \int g(\Phi_0, 0, x) d\Phi_0(x) \right] = \mathbb{E} \left[ \int g(\Phi_0, x, 0) d\Phi_0(x) \right].
\]

- This equation characterizes **mass-stationary** random measures.
5. Scaling limits

- Assume \([G_n, o_n, \mu_n]\) is such that
  - \(G_n\): A finite metric space,
  - \(o_n \in G_n\) chosen uniformly at random,
  - \(\mu_n\): The counting measure on \(G_n\).

- Assume \([\epsilon_n G_n, o_n, \delta_n \mu_n]\) converges weakly.

- Example:
  - \(\mathbb{Z}^d \Rightarrow \mathbb{R}^d\).
  - Random trees \(\Rightarrow\) Brownian continuum random tree.
  - Zeros of simple random walk \(\Rightarrow\) Zeros of Brownian motion.
  - Cayley graph \(\Rightarrow\) A locally-compact group.

- We will see that there exists an MTP for the scaling limit.
The Goals

Our goals:

- A unification of the various versions of the MTP.
- Generalizing Palm theory in order to use for studying the dimension of scaling limits.
1 The Mass Transport Principle in Various Subjects

2 Unimodular Continuum Spaces

3 Palm Theory
\( \mathcal{M}_*: = \) The space of all \((X, o, \mu)\), where:
- \(X\) is a metric space (and is \textit{boundedly-compact}),
- \(o \in X\) (the root),
- \(\mu\) is a measure on \(X\) (and is \textit{boundedly-finite}).

\(\mathcal{M}_*\) is a Polish space (with the GHP metric).

A random \textit{rmm} space (rooted measured metric space):
A random element \([X, o, \mu]\) in \(\mathcal{M}_*\).

\[
\mathbb{E}[f(X, o, \mu)] = \int_{\mathcal{M}_*} f([X, o, \mu]) d\mathbb{P}([X, o, \mu]).
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\( M_* := \) The space of all \((X, o, \mu)\), where:
- \( X \) is a metric space (and is \textit{boundedly-compact}),
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\( M_* \) is a Polish space (with the GHP metric).

A \textbf{random rmm space} (rooted measured metric space):
A random element \([X, o, \mu]\) in \( M_* \).

\[
\mathbb{E}[f(X, o, \mu)] = \int_{M_*} f([X, o, \mu]) d\mathbb{P}([X, o, \mu]).
\]
\( M_{**} := \text{The space of all } (X, o, p, \mu). \)

- \( p \in X \) is called the second root.

- \([X, o, \mu] \): A random rmm space.

**Definition**

\([X, o, \mu] \) is a **unimodular random rmm space** if for all \( g \):

\[
\mathbb{E} \left[ \int_X g(o, x) d\mu(x) \right] = \mathbb{E} \left[ \int_X g(x, o) d\mu(x) \right],
\]

where \( g(o, x) := g(X, o, x, \mu) \) and \( g : M_{**} \rightarrow \mathbb{R}_{\geq 0} \) is measurable.

\[
\mathbb{E} [g^+(o)] = \mathbb{E} [g^-(o)]
\]
Trivial Examples

- When $\mu = 0$.
- When $\mu = \delta_o$.
- Compact spaces:
  - $[X, \mu]$: Any random compact measured metric space, $o \in X$ random with distribution proportional to $\mu$,
  - Then $[X, o, \mu]$ is unimodular.
- Compact unimodular spaces are re-rooting invariant.
- In general, heuristically, the root is a typical point.
Trivial Examples

- When $\mu = 0$.
- When $\mu = \delta_o$.
- Compact spaces:
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- Compact unimodular spaces are re-rooting invariant.
- In general, heuristically, the root is a typical point.
Examples

- (Palm version of) Stationary point processes,
  - \([\Phi_0, 0, \text{counting}(\Phi_0)]\).
  - \([\mathbb{R}^d, 0, \text{counting}(\Phi_0)]\). (\(\rightarrow\) no need to have \(\text{supp}(\mu) = X\))

- Point-stationary point processes,
- Unimodular random graphs,
- Unimodular discrete spaces,
- (Palm version of) Stationary random measures,
- Mass-stationary random measures.
- Unimodular random manifolds (Abért and Biringer, 22).
Examples: Weak Limits

Lemma

Any weak limit of a sequence of unimodular spaces is unimodular.

Corollary

Scaling limits are unimodular (under the assumptions already mentioned).

Corollary

All compact scaling limits have the re-rooting invariance property:
If $o' \in X$ is random with distribution proportional to $\mu$, then $[X, o', \mu] \sim [X, o, \mu]$. 
Some symmetric spaces are unimodular:

- \([\mathbb{R}^d, 0, \text{Leb}]\) and \([\mathbb{H}^d, o, \text{vol}]\).
- Every *unimodular* topological group (i.e., when the left and right Haar measures are equal).
- Every symmetric metric space (or manifold) with a unimodular symmetry group (e.g., \(\mathbb{H}^n\) or \(\mathbb{S}^n\)),
  - or having an action of a unimodular group that is transitive and measure preserving.
General Properties

- If $S \subseteq X$ is an equivariant random subset, then
  \[ \mu(S) > 0 \iff \mathbb{P}[o \in S] > 0. \]

- Invariance under changing the root according to a random walk (generalization of Mecke’s stationarity under bijective point shifts).
- Ergodic decomposition.
- Equivalence of amenability and hyperfiniteness.
Outline

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Random Measures

- \([X, o, \mu, \Phi]\) is a random element in \(\mathcal{M}_*^2\), where

\[
\mathcal{M}_*^2 := \{(X, o, \mu, \varphi) : \varphi \text{ is a measure on } X\}.
\]

- Assume \([X, o, \mu, \Phi]\) is unimodular; i.e., the MTP holds when \(g\) depends on \(\Phi\) as well.

- Equivalently:
  - First, sample \([X, o, \mu]\),
  - Then choose a random \(\Phi \in \mathcal{M}(X)\) such that its distribution does not depend on \(o\) and is isomorphism-invariant.

**Definition**

We say that \(\Phi\) is an equivariant random measure on \([X, o, \mu]\).
Random Measures

- $[X, o, \mu, \Phi]$ is a random element in $\mathcal{M}_2^\star$, where

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**Definition**

We say that \(\Phi\) is an **equivariant random measure on** \([X, o, \mu]\).
Examples

1. \( \Phi = \mu \) or any factor of \((X, \mu)\).

2. The intensity measure of \( \Phi \) is also an equivariant (factor) measure: 
   \[ \lambda(X, \mu) := \mathbb{E} [\Phi(X, \mu)] \]

3. \( \Phi := \) the Poisson point process with intensity measure \( \mu \).
Examples

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Towards Palm

- There is no $[0, 1]^d$ here!
The classical definition does not generalize.
- Other definitions of Palm:
  1. via the Campbell measure.
  2. via a tessellation.
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- Other definitions of Palm:
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  2. via a tessellation.
Palm Via Tessellation

- $\Phi$: A stationary point process in $\mathbb{R}^d$.
- **Equivariant tessellation**: Assigning a cell to each point of $\Phi$ equivariantly.

- **Fair tessellation**: When all cells have equal volumes.

**Theorem**

*If the cell of $x \in \Phi$ contains 0, then $\Phi - x \sim \Phi_0$.***
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*If the cell of $x \in \Phi$ contains 0, then $\Phi - x \sim \Phi_0$.***
A Generalization

- Assume a function $h(x, y) \geq 0$ is a function depending on $\Phi$ (as a factor of $\Phi$) such that

$$\forall y \in \Phi : h^-(y) := \int h(x, y) dx = 1.$$ 

- Example: Given a fair tessellation, let $h(x, y) := \lambda$ if $x \in \text{cell}(y)$.

**Theorem**

Palm of $\Phi$ is obtained by a biasing and shifting the origin to a point of $\Phi$ chosen with distribution proportional to $h(0, \cdot)$; i.e.,

$$P [\Phi_0 \in A] = \frac{1}{\lambda} E \left[ \sum_{y \in \Phi} 1_A(\Phi - y) h(0, y) \right],$$

where $\lambda$ is the intensity of $\Phi$. 
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**Theorem**

*Palm of $\Phi$ is obtained by a biasing and shifting the origin to a point of $\Phi$ chosen with distribution proportional to $h(0, \cdot)$; i.e.,*

$$\mathbb{P} [\Phi_0 \in A] = \frac{1}{\lambda} \mathbb{E} \left[ \sum_{y \in \Phi} 1_A(\Phi - y) h(0, y) \right],$$

*where $\lambda$ is the intensity of $\Phi$.*
Assume $h : \mathcal{M}^{2}_{**} \to \mathbb{R}^{\geq 0}$ is such that for all $(X, y, \mu, \varphi)$,

$$h^{-}(y) := \int_{\mathcal{X}} h(x, y) d\mu(x) = 1 \quad (\text{if } \mu \neq 0).$$

Bias and choose a new root $\sim h(o, \cdot) \Phi$; i.e.,

**Definition**

Define a measure $Q$ on $\mathcal{M}^{2}_{*}$ by:

$$Q(A) := \mathbb{E} \left[ \int_{\mathcal{X}} 1_{A}(X, y, \mu, \Phi) h(o, y) d\Phi(y) \right].$$

Define the **intensity** of $\Phi$ by $\lambda := |Q| = Q(\mathcal{M}^{2}_{*})$. Define the probability measure $P_{0} := \frac{1}{\lambda} Q$ (if $0 < \lambda < \infty$). $P_{0}$ is the distribution of the **Palm version**.
Assume \( h : \mathcal{M}_\bullet^2 \rightarrow \mathbb{R}_{\geq 0} \) is such that for all \( (X, y, \mu, \varphi) \),

\[
h^-(y) := \int_X h(x, y) \, d\mu(x) = 1 \quad \text{(if } \mu \neq 0\text{)}.
\]

Bias and choose a new root \( \sim h(o, \cdot)\Phi \); i.e.,

**Definition**
Define a measure \( Q \) on \( \mathcal{M}_\bullet^2 \) by:

\[
Q(A) := \mathbb{E} \left[ \int_X 1_A(X, y, \mu, \Phi) h(o, y) \, d\Phi(y) \right].
\]

Define the **intensity** of \( \Phi \) by \( \lambda := |Q| = Q(\mathcal{M}_\bullet^2) \).
Define the probability measure \( P_0 := \frac{1}{\lambda} Q \) (if \( 0 < \lambda < \infty \)).

\( P_0 \) is the distribution of the **Palm version**.
Theorem (Campbell Formula)

For all measurable functions $g \geq 0$ on $\mathcal{M}_**^2$, by denoting $g(x, y) := g(X, x, y, \mu, \Phi)$,

$$
\mathbb{E} \left[ \int_X g(o, y) d\Phi(y) \right] = \lambda \mathbb{E}_0 \left[ \int_X g(x, o) d\mu(x) \right].
$$

In addition, $\mathbb{P}_0$ is the unique probability measure on $\mathcal{M}_*^2$ with this property.

**Corollary.** Palm does not depend on the choice of $h$. 

Unimodularity of Palm

- \([X, o, \mu, \Phi]\) unimodular.

**Lemma**

Under \(\mathbb{P}_0\), \([X, o, \Phi]\) is unimodular, and so is \([X, o, \Phi, \mu]\).

**Corollary**

Under \(\mathbb{P}_0\), the Palm of \(\mu\) (as random measure on \([X, o, \Phi]\)) is \(\mathbb{P}\).

\[\text{Palm inversion} = \text{Palm}\]
Unimodularity of Palm

- \([X, o, \mu, \Phi]\) unimodular.

**Lemma**

Under \(P_0\), \([X, o, \Phi]\) is unimodular, and so is \([X, o, \Phi, \mu]\).

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\(\text{Palm inversion} = \text{Palm}\)
• \([X, o, \mu, \Phi]\) unimodular.

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*Under \(P_0\), \([X, o, \Phi]\) is unimodular, and so is \([X, o, \Phi, \mu]\).*

**Corollary**

*Under \(P_0\), the Palm of \(\mu\) (as random measure on \([X, o, \Phi]\)) is \(P\).*

*Palm inversion = Palm*
Examples

- If $\Phi = \mu|_S$, where $S$ is a factor subset,
  - Palm = conditioning on $o \in S$.
- If $\Phi$ is the Poisson point process with intensity measure $c\mu$,
  - Palm version is $\Phi \cup \{o\}$.
- Planar Duals:
  - To make the dual $G'$ of $G$ unimodular:
    - $X := G \cup G'$,
    - $\mu :=$ the counting measure of $G$,
    - $\Phi :=$ the counting measure of $G'$,
    - it is enough to consider the Palm of $\Phi$.
- Adding vertices and edges to a unimodular graph (unimodular extension) is an instance of Palm.
Thank you!
Subset Selection

Two equivalent definitions:

1. If $A \subseteq \mathcal{M}_*$ is measurable, then $S := S(X, \mu) := \{ y \in X : (X, y, \mu) \in A \}$ is called a **factor subset**.

2. A **factor subset** is a map $(X, \mu) \mapsto S(X, \mu) \subseteq X$ such that it is isometry-equivariant and $A := \{(X, y, \mu) : y \in S(X, \mu)\}$ is measurable.

**Lemma (Everything Happens at the Root)**

If $[X, o, \mu]$ is unimodular and $S$ is a factor subset, then:

\[
\begin{align*}
  o \in S & \quad \text{a.s.} \iff \mu(X \setminus S) = 0 \quad \text{a.s.}, \\
  \mathbb{P}[\mu(S) > 0] > 0 & \iff \mathbb{P}[o \in S] > 0.
\end{align*}
\]

**Corollary**

\[
  o \in \text{supp}(\mu) \quad \text{a.s.}
\]
Subset Selection

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If $[X, o, \mu]$ is unimodular and $S$ is a factor subset, then:

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o \in S \text{ a.s. } \iff \mu(X \setminus S) = 0 \text{ a.s. }, \]
\[
\mathbb{P}[\mu(S) > 0] > 0 \iff \mathbb{P}[o \in S] > 0.
\]

**Corollary**

$o \in \text{supp}(\mu) \text{ a.s.}$
[\mathcal{X}, \mathcal{O}, \mu]$: unimodal

Assume for each \((\mathcal{X}, \mathcal{O}, \mu)\), a probability measure \(\alpha = \alpha(\mathcal{X}, \mathcal{O}, \mu)\) on \(\mathcal{X}\) is given (isometry-equivariant with some measurability property).

Let \(\mathcal{O}' \in \mathcal{X}\) be chosen with distribution \(\alpha\).

**Lemma**

1. \([\mathcal{X}, \mathcal{O}', \mu] \sim [\mathcal{X}, \mathcal{O}, \mu]\) if \(\mu\) is a stationary measure for the Markovian kernel on \(\mathcal{X}\).

2. This holds if \(f(\mathcal{O}, x)\) is the density of \(\alpha\) w.r.t. \(\mu\) at \(x\) and \(f^-(\mathcal{O}) = 1\) a.s., where \(f^-(\mathcal{O}) := \int_{\mathcal{X}} f(y, \mathcal{O}) \mu(dy)\).

3. If \(f(\mathcal{O}, x)\) is the density of \(\alpha\) w.r.t. \(\mu\) at \(x\) (if exists), then the density of \([\mathcal{X}, \mathcal{O}', \mu]\) w.r.t. \([\mathcal{X}, \mathcal{O}, \mu]\) is \(f^-(\mathcal{O})\).

This generalizes Mecke’s theorem (invariance under bijective point-shifts).
[\mathbf{X}, o, \mu]$: unimodular

Assume for each $(\mathbf{X}, o, \mu)$, a probability measure $\alpha = \alpha(\mathbf{X}, o, \mu)$ on $\mathbf{X}$ is given (isometry-equivariant with some measurability property).

Let $o' \in \mathbf{X}$ be chosen with distribution $\alpha$.

**Lemma**

1. $[\mathbf{X}, o', \mu] \sim [\mathbf{X}, o, \mu]$ if $\mu$ is a stationary measure for the Markovian kernel on $\mathbf{X}$.

2. This holds if $f(o, x)$ is the density of $\alpha$ w.r.t. $\mu$ at $x$ and $f^{-1}(o) = 1$ a.s., where $f^{-1}(o) : = \int_{\mathbf{X}} f(y, o) \mu(dy)$.

3. If $f(o, x)$ is the density of $\alpha$ w.r.t. $\mu$ at $x$ (if exists), then the density of $[\mathbf{X}, o', \mu]$ w.r.t. $[\mathbf{X}, o, \mu]$ is $f^{-1}(o)$.

This generalizes Mecke’s theorem (invariance under bijective point-shifts).
Random Walk

- Fix $h_0$ such that $h_0^+(\cdot) = 1$ and $h > 0$.

$$h(x, y) := \int_X \frac{h_0(x, z)h_0(y, z)}{h_0^-(z)} \, d\mu(z).$$

- So, $h^+(\cdot) = h^-(\cdot) = 1$.

- Let $[X, o, \mu]$ be random.

- Define a random walk $(x_n)_n$ on $X$ such that $x_0 = o$ and $x_{n+1} \sim h(x_n, \cdot) \mu$.

Theorem

$[X, o, \mu]$ is unimodular if and only if $(x_n)_n$ is stationary and reversible; i.e.,

$$[X, x_1, \mu, (x_{n+1})_n] \sim [X, o, \mu, (x_n)_n],$$

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Ergodicity

- An event $A$ is **(root-change-) invariant** if $[X, o, \mu] \in A \implies [X, y, \mu] \in A, \forall y \in X$.

- **Definition**: A unimodular rmm space $[X, o, \mu]$ is ergodic when $\mathbb{P}[A] \in \{0, 1\}$ for every invariant event $A$.

---

**Theorem (Ergodic Decomposition)**

- $[X, o, \mu]$ is ergodic if and only if the random walk $(x_n)$ is ergodic.

- Every unimodular probability measure can be uniquely written as a mixture of ergodic probability measures.
An event $A$ is **(root-change-) invariant** if $[X, o, \mu] \in A \Rightarrow [X, y, \mu] \in A, \forall y \in X$.

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- $[X, o, \mu]$ is ergodic if and only if the random walk $(x_n)$ is ergodic.

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Let $[X, o, \mu]$ be unimodular.

**Theorem (Amenability)**

The following are equivalent:

(i) There exists a local mean.

(ii) There exists an approximate mean.

(iii) Hyperfiniteness.

(iv) Folner condition.
To (almost) every \((X, o, \mu)\), assign a map \(m : L^\infty(X, \mu) \to \mathbb{R}\) such that:

- \(m\) is a positive linear functional.
- \(m\) is isomorphism-invariant.
- \(\forall y \in X : m(x, o, \mu) = m(x, y, \mu)\).
- Some measurability condition.

**Definition:** This is called a **Local mean**.

To (almost) every \((X, o, \mu)\), assign a sequence \(\lambda_n : X \to \mathbb{R} \geq 0\) such that:

- \(\lambda_n\) is isomorphism-invariant and measurable.
- \(\forall y \in X : \int_X \lambda_n(y, \cdot) d\mu = 1\) a.s.
- \(\forall y \in X : \|\lambda_n(o, \cdot) - \lambda_n(y, \cdot)\|_1 \to 0\) a.s.

**Definition:** This is called an **approximate mean**.
To (almost) every \((X, o, \mu)\), assign a map \(m : L^\infty(X, \mu) \to \mathbb{R}\) such that:

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**Definition:** This is called an **approximate mean**.
To (almost) every \((X, o, \mu)\), assign a partition \(\Pi\) of \(X\) such that it is invariant, measurable, and every element of \(\Pi\) has finite mass w.r.t. \(\mu\).

Allow \(\Pi\) to be random; e.g., depending on a random measure on \((X, o, \mu)\).

**Definition:** This is called an **equivariant finite partition**.

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**Definition (Hyperfiniteness)**

Three definitions:

1. \(\exists\) nested equivariant finite partitions \(\Pi_n\) s.th. \(\mathbb{P} \left[ \bigcup_n \Pi_n(o) = X \right] = 1\).
2. \(\exists\) nested equivariant finite partitions \(\Pi_n\) s.th.
   \(\forall r < \infty : \mathbb{P} \left[ \exists n : B_r(o) \subseteq \Pi_n(o) \right] = 1\).
3. \(\forall r < \infty, \forall \epsilon > 0, \exists\) an equivariant finite partition \(\Pi\) s.th.
   \(\mathbb{P} \left[ B_r(o) \nsubseteq \Pi(o) \right] < \epsilon\).
Folner Condition

Definition

Two definitions:

1. \( \forall r < \infty, \forall \epsilon > 0, \exists \), an equivariant finite partition \( \Pi \) s.th.

\[
E \left[ \frac{\mu(\partial_r \Pi(o))}{\mu(\Pi(o))} \right] < \epsilon.
\]

2. \( \exists \) equivariant nested finite partitions \( \Pi_n \) s.th.

\[
\forall r : \frac{\mu(\partial_r \Pi_n(o))}{\mu(\Pi_n(o))} \to 0, \quad a.s.
\]
Proof Method

- Let $\Phi$ be the marked Poisson point process on $X$ with intensity measure $\mu$.
- Consider the Palm version of $\Phi$.
- This gives a countable Borel equivalence relation $R$ and the Palm distribution is an invariant measure.
- We prove that each definition is equivalent to the analogous definition for $R$.
- We use the amenability theorem for Borel equivalence relations.
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