Unimodular Continuum Spaces

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Outline

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Introduction

Connection between various fields:

- Stationary point processes,
- Unimodular random graphs,
- Unimodular discrete spaces,
- Stationary random measures,
- Scaling limits,
- Borel equivalence relations.

Key property: The **mass transport principle (MTP)**.

1. Point Processes

- Φ: A stationary point process on $\mathbb{R}^d$.
  - i.e., a random discrete subset of $\mathbb{R}^d$,
  - s.th., $\forall t \in \mathbb{R}^d : \Phi + t \sim \Phi$.
- The **Palm version** of Φ:
  - $\Phi_0 := \Phi$ conditioned on containing 0,
  - or Φ seen from a *typical point of* Φ.
  - Formally:

$$\mathbb{E}\left[h(\Phi_0)\right] = \frac{1}{\lambda}\mathbb{E}\left[\sum_{x \in \Phi \cap [0,1]^d} h(\Phi - x)\right].$$

- **Mecke's formula**:
  For all measurable functions $h(\Phi_0, x) \geqslant 0$ (for $x \in \mathbb{R}^d$):
  $\mathbb{E}\left[\sum_{x\in\Phi_0} h(\Phi_0, x)\right] = \mathbb{E}\left[\sum_{x\in\Phi_0} h(\Phi_0 - x, -x)\right].$
- Let $g(\Phi_0, x, y) := h(\Phi_0 - x, y - x) \Rightarrow$

Theorem (MTP)

For all measurable functions $g(\Phi_0, x, y) \geqslant 0$ that are translation-invariant:

$$\mathbb{E}\left[\sum_{x\in\Phi_0} g(\Phi_0, 0, x)\right] = \mathbb{E}\left[\sum_{x\in\Phi_0} g(\Phi_0, x, 0)\right].$$

2. Unimodular Graphs

- $\mathcal{G}_*$: The space of all rooted graphs (G, o) $(o \in V(G))$ up to isomorphisms.
- $[\boldsymbol{G}, \boldsymbol{o}]$: A random rooted graph.
- It is called **unimodular** if

$$\mathbb{E}\left[\sum_{x \in \boldsymbol{G}} g(\boldsymbol{G}, \boldsymbol{o}, x)\right] = \mathbb{E}\left[\sum_{x \in \boldsymbol{G}} g(\boldsymbol{G}, x, \boldsymbol{o})\right] \qquad \text{(MTP)}$$

  for all measurable functions $g(G, x, y) \geqslant 0$ (for $x, y \in V(G)$) that are isometry-invariant.
- **Example**:
  1. Every finite graph G with a uniformly-random root $\boldsymbol{o} \in V(G)$.
  2. Cayley graphs.
  3. **Example**: Any graph constructed *equivariantly* on (the Palm version of) a stationary point process.

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3. Unimodular Discrete Spaces

- $[\boldsymbol{D}, \boldsymbol{o}]$: A random rooted discrete metric space.
  - $\boldsymbol{D}$ should be *boundedly-finite*.
- It is called **unimodular** if for all measurable functions $g(D, x, y) \geqslant 0$ (for $x, y \in D$) that are isometry-invariant,

$$\mathbb{E}\left[\sum_{x\in\boldsymbol{D}} g(\boldsymbol{D}, \boldsymbol{o}, x)\right] = \mathbb{E}\left[\sum_{x\in\boldsymbol{D}} g(\boldsymbol{D}, x, \boldsymbol{o})\right]. \qquad \text{(MTP)}$$

- (Almost-) Unification of:
  - Unimodular graphs,
  - Palm version of stationary point processes,
  - Point-stationary point processes.

4. Random Measures

- Φ: A stationary random measure on $\mathbb{R}^d$.
  - i.e., $\forall t \in \mathbb{R}^d : \Phi + t \sim \Phi$.
- **Example**: Every point process is a random measure.
- Φ_0: The **Palm version** of Φ,
  - or Φ seen from a *typical point*.
  - Heuristically:

$$\mathbb{E}\left[g(\Phi_0, 0)\right] \longleftrightarrow \int g(\Phi, x) d\Phi(x).$$

Theorem (MTP)

For all measurable functions $g(\Phi_0, x, y) \geqslant 0$ that are translation-invariant:

$$\mathbb{E}\left[\int g(\Phi_0, 0, x) d\Phi_0(x)\right] = \mathbb{E}\left[\int g(\Phi_0, x, 0) d\Phi_0(x)\right].$$

- This equation characterizes **mass-stationary** random measures.

5. Scaling limits

- Assume $[\boldsymbol{G}_n, \boldsymbol{o}_n, \boldsymbol{\mu}_n]$ is such that
  - $\boldsymbol{G}_n$: A finite metric space,
  - $\boldsymbol{o}_n \in G_n$ chosen uniformly at random,
  - $\boldsymbol{\mu}_n$: The counting measure on G_n.
- Assume $[\epsilon_n G_n, o_n, \delta_n \mu_n]$ converges weakly.
- Example:
  - $\mathbb{Z}^d \Rightarrow \mathbb{R}^d$.
  - Random trees $\Rightarrow$ Brownian continuum random tree.
  - Zeros of simple random walk $\Rightarrow$ Zeros of Brownian motion.
  - Cayley graph $\Rightarrow$ A locally-compact group.
- We will see that there exists an MTP for the scaling limit.

The Goals

Our goals:

- A unification of the various versions of the MTP.
- Generalizing Palm theory in order to use for studying the dimension of scaling limits.

Outline

Random Continuum Spaces

- $\mathcal{M}_*$:= The space of all (X, o, μ) , where:
  - X is a metric space (and is *boundedly-compact*),
  - $o \in X$ (the root),
  - μ is a measure on X (and is *boundedly-finite*).
- $\mathcal{M}_*$ is a Polish space (with the GHP metric).
- A **random rmm space** (rooted measured metric space): A random element $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ in $\mathcal{M}_*$.

$$\mathbb{E}\left[f(\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu})\right] = \int_{\mathcal{M}_*} f([X, o, \mu]) d\mathbb{P}([X, o, \mu]).$$

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Unimodular Continuum Spaces

- $\mathcal{M}_{**} :=$ The space of all (X, o, p, μ).
  - $p \in X$ is called the second root.
- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$: A random rmm space.

Definition

$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is a **unimodular random rmm space** if for all g:

$$\mathbb{E}\left[\int_{\boldsymbol{X}} g(\boldsymbol{o}, x) d\boldsymbol{\mu}(x)\right] = \mathbb{E}\left[\int_{\boldsymbol{X}} g(x, \boldsymbol{o}) d\boldsymbol{\mu}(x)\right],$$

where $g(\boldsymbol{o}, x) := g(\boldsymbol{X}, \boldsymbol{o}, x, \boldsymbol{\mu})$ and $g : \mathcal{M}_{**} \to \mathbb{R}^{\geqslant 0}$ is measurable.

$$\mathbb{E}\left[g^{+}(\boldsymbol{o})\right] = \mathbb{E}\left[g^{-}(\boldsymbol{o})\right]$$

Trivial Examples

- When $\boldsymbol{\mu} = 0$.
- When $\boldsymbol{\mu} = \delta_{\boldsymbol{o}}$.
- Compact spaces:
  - $[\boldsymbol{X}, \boldsymbol{\mu}]$: Any random compact measured metric space,
  - $\boldsymbol{o} \in \boldsymbol{X}$ random with distribution proportional to $\boldsymbol{\mu}$,
  - Then $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is unimodular.
- Compact unimodular spaces are re-rooting invariant.
- In general, heuristically, the root is a *typical point*.

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Examples

- (Palm version of) Stationary point processes,
  - $[\Phi_0, 0, \mathrm{counting}(\Phi_0)]$.
  - $[\mathbb{R}^d, 0, \mathrm{counting}(\Phi_0)]$. ($\rightarrow$ no need to have $\mathrm{supp}(\boldsymbol{\mu}) = \boldsymbol{X}$)
- Point-stationary point processes,
- Unimodular random graphs,
- Unimodular discrete spaces,
- (Palm version of) Stationary random measures,
- Mass-stationary random measures.
- Unimodular random manifolds (Abért and Biringer, 22).

Examples: Weak Limits

Lemma

Any weak limit of a sequence of unimodular spaces is unimodular.

Corollary

Scaling limits are unimodular (under the assumptions already mentioned).

Corollary

All compact scaling limits have the ***re-rooting invariance property****:*
If $\boldsymbol{o}' \in \boldsymbol{X}$ *is random with distribution proportional to* $\boldsymbol{\mu}$*, then*
$[\boldsymbol{X}, \boldsymbol{o}', \boldsymbol{\mu}] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$.

Examples: Symmetric Spaces

Some symmetric spaces are unimodular:

- $[\mathbb{R}^d, 0, \mathrm{Leb}]$ and $[\mathbb{H}^d, o, \mathrm{vol}]$.
- Every *unimodular* topological group (i.e., when the left and right Haar measures are equal).
- Every symmetric metric space (or manifold) with a unimodular symmetry group (e.g., $\mathbb{H}^n$ or $\mathbb{S}^n$),
  - or having an action of a unimodular group that is transitive and measure preserving.

General Properties

- If $\boldsymbol{S} \subseteq \boldsymbol{X}$ is an equivariant random subset, then

$$\boldsymbol{\mu}(\boldsymbol{S}) > 0 \iff \mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}] > 0.$$

- Invariance under changing the root according to a random walk (generalization of Mecke's stationarity under bijective point shifts).
- Ergodic decomposition.
- Equivalence of amenability and hyperfiniteness.

Outline

Random Measures

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ is a random element in $\mathcal{M}_*^2$, where

$$\mathcal{M}_*^2 := \{(X, o, \mu, \varphi) : \varphi \text{ is a measure on } X\}.$$

- Assume $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ is unimodular; i.e., the MTP holds when g depends on Φ as well.
- Equivalently:
  - First, sample $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$,
  - Then choose a random $\Phi \in \mathcal{M}(\boldsymbol{X})$ such that its distribution does not depend on $\boldsymbol{o}$ and is isomorphism-invariant.

Definition

We say that Φ is an **equivariant random measure on** $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$.

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We say that Φ is an **equivariant random measure on** $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$.

Examples

1. $\Phi = \mu$ or any factor of (X, μ).
2. The **intensity measure** of Φ is also an equivariant (factor) measure: $\lambda(X, \mu) := \mathbb{E}\left[\Phi(X, \mu)\right]$.
3. $\Phi :=$ the **Poisson point process** with intensity measure μ.

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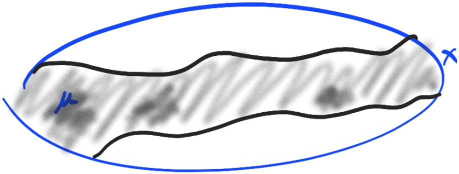

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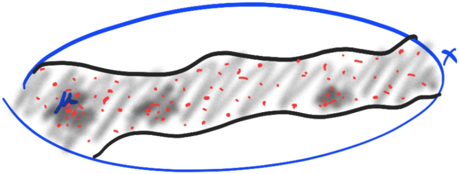

Towards Palm

- There is no $[0, 1]^d$ here!
  The classical definition does not generalize.
- Other definitions of Palm:
  1. via the Campbell measure.
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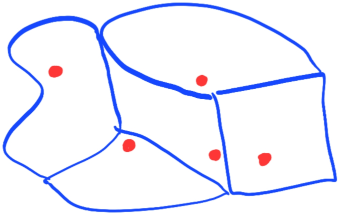
Palm Via Tessellation

- Φ: A stationary point process in $\mathbb{R}^d$.
- **Equivariant tessellation**: Assigning a cell to each point of Φ equivariantly.



- **Fair tessellation**: When all cells have equal volumes.

Theorem

If the cell of $\boldsymbol{x} \in \Phi$ contains 0, then $\Phi - \boldsymbol{x} \sim \Phi_0$.

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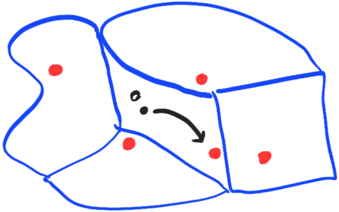

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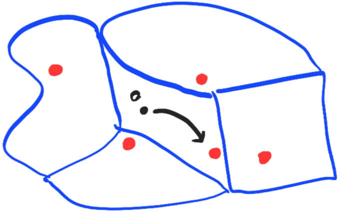

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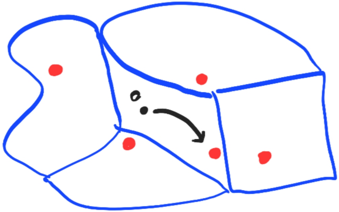

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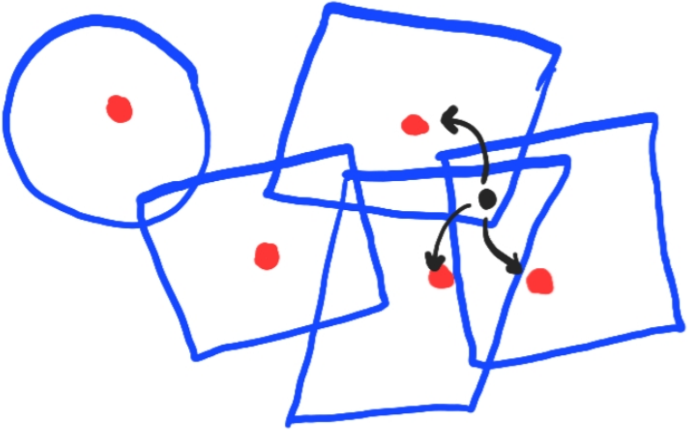

A Generalization

- Assume a function $h(x, y) \geqslant 0$ is a function depending on Φ (as a factor of Φ) such that

$$\forall y \in \Phi : h^{-}(y) := \int h(x, y) dx = 1.$$

- Example: Given a fair tessellation, let $h(x, y) := \lambda$ if $x \in \mathrm{cell}(y)$.

Theorem

Palm of Φ is obtained by a biasing and shifting the origin to a point of Φ chosen with distribution proportional to $h(0, \cdot)$; i.e.,

$$\mathbb{P}\left[\Phi_0 \in A\right] = \frac{1}{\lambda}\mathbb{E}\left[\sum_{y\in\Phi} 1_A(\Phi - y)h(0, y)\right],$$

where λ is the intensity of Φ.

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Palm on Unimodular Spaces

- Assume $h : \mathcal{M}^2_{**} \to \mathbb{R}^{\geqslant 0}$ is such that for all (X, y, μ, φ),

$$h^-(y) := \int_X h(x, y) d\mu(x) = 1 \quad (\text{if } \mu \neq 0).$$

- Bias and choose a new root $\sim h(\boldsymbol{o}, \cdot)\Phi$; i.e.,

Definition

Define a measure Q on $\mathcal{M}^2_*$ by:

$$Q(A) := \mathbb{E}\left[\int_{\boldsymbol{X}} 1_A(\boldsymbol{X}, y, \boldsymbol{\mu}, \Phi) h(\boldsymbol{o}, y) d\Phi(y)\right].$$

Define the **intensity** of Φ by $\lambda := |Q| = Q(\mathcal{M}^2_*)$.
Define the probability measure $\mathbb{P}_0 := \frac{1}{\lambda} Q$ (if $0 < \lambda < \infty$).
$\mathbb{P}_0$ is the distribution of the **Palm version**.

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Palm Via the Campbell Measure

Theorem (Campbell Formula)

For all measurable functions $g \geqslant 0$ *on* $\mathcal{M}^2_{**}$, *by denoting* $g(x, y) := g(\boldsymbol{X}, x, y, \boldsymbol{\mu}, \Phi)$,

$$\mathbb{E}\left[\int_{\boldsymbol{X}} g(\boldsymbol{o}, y) d\Phi(y)\right] = \lambda \mathbb{E}_0\left[\int_{\boldsymbol{X}} g(x, \boldsymbol{o}) d\boldsymbol{\mu}(x)\right].$$

In addition, $\mathbb{P}_0$ *is the unique probability measure on* $\mathcal{M}^2_*$ *with this property.*

- **Corollary**. Palm does not depend on the choice of h.

Unimodularity of Palm

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ unimodular.

Lemma

Under $\mathbb{P}_0$, $[\boldsymbol{X}, \boldsymbol{o}, \Phi]$ is unimodular, and so is $[\boldsymbol{X}, \boldsymbol{o}, \Phi, \boldsymbol{\mu}]$.

Corollary

Under $\mathbb{P}_0$, the Palm of $\boldsymbol{\mu}$ (as random measure on $[\boldsymbol{X}, \boldsymbol{o}, \Phi]$) is $\mathbb{P}$.

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Examples

- If $\Phi = \boldsymbol{\mu}|_S$, where S is a factor subset,
  - Palm = conditioning on $\boldsymbol{o} \in S$.
- If Φ is the Poisson point process with intensity measure $c\boldsymbol{\mu}$,
  - Palm version is $\Phi \cup \{\boldsymbol{o}\}$.
- Planar Duals:
  - $[\boldsymbol{G}, \boldsymbol{o}]$: a unimodular planar graph.
  - To make the dual $\boldsymbol{G}'$ of $\boldsymbol{G}$ unimodular:
    - $\boldsymbol{X} := \boldsymbol{G} \cup \boldsymbol{G}'$,
    - $\boldsymbol{\mu} :=$ the counting measure of $\boldsymbol{G}$,
    - $\Phi :=$ the counting measure of $\boldsymbol{G}'$,
    - it is enough to consider the Palm of Φ.
- Adding vertices and edges to a unimodular graph (unimodular extension) is an instance of Palm.

Thank you!

Subset Selection

Two equivalent definitions:

1. If $A \subseteq \mathcal{M}_*$ is measurable, then $S := S(X, \mu) := \{y \in X : (X, y, \mu) \in A\}$ is called a **factor subset**.
2. A **factor subset** is a map $(X, \mu) \mapsto S(X, \mu) \subseteq X$ such that it is isometry-equivariant and $A := \{(X, y, \mu) : y \in S(X, \mu)\}$ is measurable.

Lemma (Everything Happens at the Root)

If $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is unimodular and S is a factor subset, then:

$$\boldsymbol{o} \in S \text{ a.s.} \iff \boldsymbol{\mu}(X \setminus S) = 0 \text{ a.s. },$$
$$\mathbb{P}[\boldsymbol{\mu}(S) > 0] > 0 \iff \mathbb{P}[\boldsymbol{o} \in S] > 0.$$

Corollary

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Root-Change

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$: unimodular
- Assume for each (X, o, μ), a probability measure $\alpha = \alpha_{(X,o,\mu)}$ on X is given (isometry-equivariant with some measurability property).
- Let $\boldsymbol{o}' \in \boldsymbol{X}$ be chosen with distribution α.

Lemma

1. $[\boldsymbol{X}, \boldsymbol{o}', \boldsymbol{\mu}] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ *if* $\boldsymbol{\mu}$ *is a stationary measure for the Markovian kernel on* $\boldsymbol{X}$.
2. *This holds if* $f(o, x)$ *is the density of* α *w.r.t.* μ *at* x *and* $f^{-}(\boldsymbol{o}) = 1$ *a.s., where* $f^{-}(\boldsymbol{o}) := \int_{\boldsymbol{X}} f(y, \boldsymbol{o})\boldsymbol{\mu}(dy)$.
3. *If* $f(o, x)$ *is the density of* α *w.r.t.* μ *at* x *(if exists), then the density of* $[\boldsymbol{X}, \boldsymbol{o}', \boldsymbol{\mu}]$ *w.r.t.* $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ *is* $f^{-}(\boldsymbol{o})$.

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Random Walk

- Fix h_0 such that $h_0^+(\cdot) = 1$ and $h > 0$.

$$h(x, y) := \int_X \frac{h_0(x, z) h_0(y, z)}{h_0^-(z)} d\mu(z).$$

- So, $h^+(\cdot) = h^-(\cdot) = 1$.
- Let $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ be random.
- Define a random walk $(\boldsymbol{x}_n)_n$ on $\boldsymbol{X}$ such that $\boldsymbol{x}_0 = \boldsymbol{o}$ and $\boldsymbol{x}_{n+1} \sim h(\boldsymbol{x}_n, \cdot)\boldsymbol{\mu}$.

Theorem

$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is unimodular if and only if $(\boldsymbol{x}_n)_n$ is stationary and reversible; i.e.,

$$[\boldsymbol{X}, \boldsymbol{x}_1, \boldsymbol{\mu}, (\boldsymbol{x}_{n+1})_n] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, (\boldsymbol{x}_n)],$$
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Ergodicity

- An event A is **(root-change-) invariant** if $[X, o, \mu] \in A \Rightarrow [X, y, \mu] \in A, \forall y \in X$.
- **Definition:** A unimodular rmm space $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is ergodic when $\mathbb{P}[A] \in \{0, 1\}$ for every invariant event A.

Theorem (Ergodic Decomposition)

(i) $[\boldsymbol{X}, \boldsymbol{o}, \mu]$ *is ergodic if and only if the random walk* $(\boldsymbol{x}_n)$ *is ergodic.*

(ii) *Every unimodular probability measure can be uniquely written as a mixture of ergodic probability measures.*

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Amenability

- Let $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ be unimodular.

Theorem (Amenability)

The following are equivalent:

1. *There exists a local mean.*
2. *There exists an approximate mean.*
3. *Hyperfiniteness.*
4. *Folner condition.*

Mean

- To (almost) every (X, o, μ), assign a map $m : L^\infty(X, \mu) \to \mathbb{R}$ such that:
  - m is a positive linear functional.
  - m is isomorphism-invariant.
  - $\forall y \in X : m_{(X,o,\mu)} = m_{(X,y,\mu)}$.
  - Some measurability condition.
- **Definition:** This is called a **Local mean**.
- To (almost) every (X, o, μ), assign a sequence $\lambda_n : X \to \mathbb{R}^{\geq 0}$ such that:
  - λ_n is isomorphism-invariant and measurable.
  - $\forall y \in \boldsymbol{X} : \int_{\boldsymbol{X}} \lambda_n(y, \cdot) d\boldsymbol{\mu} = 1$ a.s.
  - $\forall y \in \boldsymbol{X} : ||\lambda_n(\boldsymbol{o}, \cdot) - \lambda_n(y, \cdot)||_1 \to 0$ a.s.
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Hyperfiniteness

- To (almost) every (X, o, μ), assign a partition Π of X such that it is invariant, measurable, and every element of Π has finite mass w.r.t. μ.
- Allow Π to be random; e.g., depending on a random measure on (X, o, μ).
- **Definition:** This is called an **equivariant finite partition**.

Definition (Hyperfiniteness)

Three definitions:

1. $\exists$ nested equivariant finite partitions Π_n s.th. $\mathbb{P}\left[\bigcup_n \Pi_n(\boldsymbol{o}) = \boldsymbol{X}\right] = 1$.
2. $\exists$ nested equivariant finite partitions Π_n s.th. $\forall r < \infty : \mathbb{P}\left[\exists n : B_r(\boldsymbol{o}) \subseteq \Pi_n(\boldsymbol{o})\right] = 1$.
3. $\forall r < \infty, \forall \epsilon > 0, \exists$ an equivariant finite partition Π s.th. $\mathbb{P}\left[B_r(\boldsymbol{o}) \nsubseteq \Pi(\boldsymbol{o})\right] < \epsilon$.

Folner Condition

Definition

Two definitions:

1. $\forall r < \infty, \forall \epsilon > 0, \exists$, an equivariant finite partition Π s.th.

$$\mathbb{E}\left[\frac{\boldsymbol{\mu}(\partial_r \Pi(\boldsymbol{o}))}{\boldsymbol{\mu}(\Pi(\boldsymbol{o}))}\right] < \epsilon.$$

2. $\exists$ equivariant nested finite partitions Π_n s.th.

$$\forall r : \frac{\boldsymbol{\mu}(\partial_r \Pi_n(\boldsymbol{o}))}{\boldsymbol{\mu}(\Pi_n(\boldsymbol{o}))} \to 0, \quad a.s.$$

Proof Method

- Let Φ be the **marked Poisson point process** on $\boldsymbol{X}$ with intensity measure $\boldsymbol{\mu}$.
- Consider the Palm version of Φ.
- This gives a countable Borel equivalence relation R and the Palm distribution is an **invariant measure**.
- We prove that each definition is equivalent to the analogous definition for R.
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